Abstract

Let $\kappa \in \text{Card}$ and $L_\kappa[X]$ be such that the fine structure theory, condensation and $\text{Card}^{L_\kappa[X]} = \text{Card} \cap \kappa$ hold. Then it is possible to prove the existence of morasses. In particular, I will prove that there is a $\kappa$-standard morass, a notion that I introduced in a previous paper. This shows the consistency of $(\omega_1, \beta)$-morasses for all $\beta \geq \omega_1$.

1 Introduction

R. Jensen formulated in the 1970’s the concept of an $(\omega_\alpha, \beta)$-morass whereby objects of size $\omega_\alpha + \beta$ could be constructed by a directed system of objects of size less than $\omega_\alpha$. He defined the notion of an $(\omega_\alpha, \beta)$-morass only for the case that $\beta < \omega_\alpha$. I introduced in a previous paper [Irr2] a definition of an $(\omega_1, \beta)$-morass for the case that $\omega_1 \leq \beta$.

This definition of an $(\omega_1, \beta)$-morass for the case that $\omega_1 \leq \beta$ seems to be an axiomatic description of the condensation property of Gödel’s constructible universe $L$ and the whole fine structure theory of it. I was, however, not able to formulate and prove this fact in form of a mathematical statement. Therefore, I defined a seemingly innocent strengthening of the notion of an $(\omega_1, \beta)$-morass, which I actually expect to be equivalent to the notion of $(\omega_1, \beta)$-morass. I call this strengthening an $\omega_1+\beta$-standard morass. As will be seen, if we construct a morass in the usual way in $L$, the properties of a standard morass hold automatically.

Using the notion of a standard morass, I was able to prove a theorem which can be interpreted as saying that standard morasses fully cover the condensation property and fine structure of $L$. More precisely, I was able to show the following [Irr2]

Theorem

Let $\kappa \geq \omega_1$ be a cardinal and assume that a $\kappa$-standard morass exists. Then there exists a predicate $X$ such that $\text{Card} \cap \kappa = \text{Card}^{L_\kappa[X]}$ and $L_\kappa[X]$ satisfies amenability, coherence and condensation.

Let me explain this. The predicate $X$ is a sequence $X = \langle X_\nu \mid \nu \in S^X \rangle$ where $S^X \subseteq \text{Lim} \cap \kappa$, and $L_\kappa[X]$ is endowed with the following hierarchy: Let $I_\nu = \langle J^X_\nu, X \mid \nu \rangle$ for $\nu \in \text{Lim} - S_1^X$ and $I_\nu = \langle J^X_\nu, X, X_\nu \rangle$ for $\nu \in S^X$ where $X_\nu \subseteq J^X_\nu$ and $J^X_0 = \emptyset$.
\[ J_{\nu+\omega}^X = \text{rud}(I_{\nu}^X) \]
\[ J_{\lambda}^X = \bigcup\{J_{\nu}^X \mid \nu \in \lambda\} \text{ for } \lambda \in \text{Lim}^2 := \text{Lim(Lim)}, \]
where \text{rud}(I_{\nu}^X) is the rudimentary closure of \( J_{\nu}^X \cup \{J_{\nu}^X\} \) relative to \( X \upharpoonright \nu \) if \( \nu \in \text{Lim} - S_{\kappa} \) and relative to \( X \upharpoonright \nu \) and \( X_{\nu} \) if \( \nu \in S_{\kappa} \). Now, the properties of \( L_{\kappa}[X] \) are defined as follows:

(Amenability) The structures \( I_{\nu} \) are amenable.

(Coherence) If \( \nu \in S_{\kappa}^X \), \( H \prec_1 I_{\nu} \) and \( \lambda = \sup(H \cap \text{On}) \), then \( \lambda \in S_{\kappa}^X \) and \( X_{\lambda} = X_{\nu} \cap J_{\lambda}^X \).

(Condensation) If \( \nu \in S_{\kappa}^X \) and \( H \prec_1 I_{\nu} \), then there is some \( \mu \in S_{\kappa}^X \) such that \( H \cong I_{\mu} \).

Moreover, if we let \( \beta(\nu) \) be the least \( \beta \) such that \( J_{\beta+\omega}^X \models \nu \) singular, then \( S_{\kappa}^X = \{\beta(\nu) \mid \nu \text{ singular in } I_{\kappa}\} \).

As will be seen, these properties suffice to develop the fine structure theory. In this sense, the theorem shows indeed what I claimed. In the present paper, I shall show the converse:

**Theorem**

If \( L_{\kappa}[X] \), \( \kappa \in \text{Card} \), satisfies condensation, coherence, amenability, \( S_{\kappa}^X = \{\beta(\nu) \mid \nu \text{ singular in } I_{\kappa}\} \) and \( \text{Card}^{L_{\kappa}[X]} = \text{Card} \cap \kappa \), then there is a \( \kappa \)-standard morass.

Since \( L \) itself satisfies the properties of \( L_{\kappa}[X] \), this also shows that the existence of \( \kappa \)-standard morasses and \( (\omega_1, \beta) \)-morasses is consistent for all \( \kappa \geq \omega_2 \) and all \( \beta \geq \omega_1 \).

Most results that can be proved in \( L \) from condensation and the fine structure theory also hold in the structures \( L_{\kappa}[X] \) of the above form. As examples, I proved in my dissertation the following two theorems whose proofs can also be seen as applications of morasses:

**Theorem**

Let \( \lambda \geq \omega_1 \) be a cardinal, \( S_{\kappa}^X \subseteq \text{Lim} \cap \lambda \), \( \text{Card} \cap \lambda = \text{Card}^{L_{\lambda}[X]} \) and \( X = \langle X_{\nu} \mid \nu \in S_{\kappa}^X \rangle \) be a sequence such that amenability, coherence, condensation and \( S_{\kappa}^X = \{\beta(\nu) \mid \nu \text{ singular in } L_{\kappa}\} \) hold. Then \( \Delta_{\kappa} \) holds for all infinite cardinals \( \kappa < \lambda \).

**Theorem**

Let \( S_{\kappa}^X \subseteq \text{Lim} \) and \( X = \langle X_{\nu} \mid \nu \in S_{\kappa}^X \rangle \) be a sequence such that amenability, coherence, condensation and \( S_{\kappa}^X = \{\beta(\nu) \mid \nu \text{ singular in } L[X]\} \) hold. Then the weak covering lemma holds for \( L[X] \). That is, if there is no non-trivial, elementary embedding \( \pi : L[X] \rightarrow L[X], \kappa \in \text{Card}^{L[X]} - \omega_2 \) and \( \tau = (\kappa^+)^{L[X]} \), then
\[ \tau < \kappa^+ \quad \Rightarrow \quad \text{cf}(\tau) = \text{card}(\kappa). \]

The present paper is a part of my dissertation [Irr1]. I thank Dieter Donder for being my adviser, Hugh Woodin for an invitation to Berkeley, where part of the work was done, and the DFG-Graduiertenkolleg “Sprache, Information, Logik” in Munich for their support.
2 The inner model $L[X]$

We say a function $f : V^n \to V$ is rudimentary for some structure $\mathcal{W} = (W, X_i)$ if it is generated by the following schemata:

- $f(x_1, \ldots, x_n) = x_i$ for $1 \leq i \leq n$
- $f(x_1, \ldots, x_n) = \{x_i, x_j\}$ for $1 \leq i, j \leq n$
- $f(x_1, \ldots, x_n) = x_i - x_j$ for $1 \leq i, j \leq n$
- $f(x_1, \ldots, x_n) = h(g_1(x_1, \ldots, x_n), \ldots, g_n(x_1, \ldots, x_n))$ where $h, g_1, \ldots, g_n$ are rudimentary
- $f(y, x_2, \ldots, x_n) = \bigcup \{g(z, x_2, \ldots, x_n) \mid z \in y\}$ where $g$ is rudimentary
- $f(x_1, \ldots, x_n) = X_i \cap x_j$ where $1 \leq j \leq n$.

**Lemma 1**

A function is rudimentary iff it is a composition of the following functions:

- $F_0(x, y) = \{x, y\}$
- $F_1(x, y) = x - y$
- $F_2(x, y) = x \times y$
- $F_3(x, y) = \{\langle u, z, v \rangle \mid z \in x \text{ and } \langle u, v \rangle \in y\}$
- $F_4(x, y) = \{\langle z, u, v \rangle \mid z \in x \text{ and } \langle u, v \rangle \in y\}$
- $F_5(x, y) = \bigcup x$
- $F_6(x, y) = \text{dom}(x)$
- $F_7(x, y) = x \cap (x \times x)$
- $F_8(x, y) = \{x[z] \mid z \in y\}$
- $F_9(x, y) = x \cap X_i$ for the predicates $X_i$ of the structure under consideration.

**Proof:** See, for example, in [Dev2].

A relation $R \subseteq V^n$ is called rudimentary if there is a rudimentary function $f : V^n \to V$ such that $R(x_i) \iff f(x_i) \neq \emptyset$.

**Lemma 2**

Every relation that is $\Sigma_0$ over the considered structure is rudimentary.

**Proof:** Let $\chi_R$ be the characteristic function of $R$. The claim follows from the facts (i)-(vi):

(i) $R \text{ rudimentary } \iff \chi_R \text{ rudimentary.}$

$\Leftarrow$ is clear. Conversely, $\chi_R = \bigcup \{g(y) \mid y \in f(x_i)\}$ where $g(y) = 1$ is constant and $R(x_i) \iff f(x_i) \neq \emptyset$.

(ii) If $R$ is rudimentary, then $\neg R$ is also rudimentary.

Since $\chi_{\neg R} = 1 - \chi_R$.

(iii) $x \in y$ and $x = y$ are rudimentary.

By $x \notin y \iff \{x\} - y \neq \emptyset$, $x \neq y \iff (x - y) \cup (y - x) \neq \emptyset$ and (ii).

(iv) If $R(y, x_i)$ is rudimentary, then $(\exists z \in y)R(z, x_i)$ and $(\forall z \in y)R(z, x_i)$ are rudimentary.
If $R(y,x_i) \Leftrightarrow f(y,x_i) \neq \emptyset$, then $(\exists z \in y) R(z,x_i) \Leftrightarrow \bigcup\{f(z,x_i) \mid z \in y\} \neq \emptyset$. The second claim follows from this by (ii).

(v) If $R_1, R_2 \subseteq V^n$ are rudimentary, then so are $R_1 \lor R_2$ and $R_1 \land R_2$.

Because $f(x,y) = x \cup y$ is rudimentary, $(R_1 \lor R_2)(x_i) \Leftrightarrow \chi_{R_1}(x_i) \cup \chi_{R_2}(x_i) \neq \emptyset$ is rudimentary. The second claim follows from that by (ii).

(vi) $x \in X_i$ is rudimentary.

Since $\{x\} \cap X_i \neq \emptyset \Leftrightarrow x \in X_i$. □

For a converse of this lemma, we define:

A function $f$ is called simple if $R(f(x_i),y_k)$ is $\Sigma_0$ for every $\Sigma_0$-relation $R(z,y_k)$.

**Lemma 3**

A function $f$ is simple iff

(i) $z \in f(x_i)$ is $\Sigma_0$

(ii) $A(z)$ is $\Sigma_0 \Rightarrow (\exists z \in f(x_i)) A(z)$ is $\Sigma_0$.

**Proof:** If $f$ is simple, then (i) and (ii) hold, because these are instances of the definition. The converse is proved by induction on $\Sigma_0$-formulas. E.g. if $R(z,y_k) :\Leftrightarrow z = y_k$, then $R(f(x_i),y_k) \Leftrightarrow f(x_i) = y_k \Leftrightarrow (\forall z \in f(x_i))(z \in y_k)$ and $(\forall z \in y_k)(z \in f(x_i))$. Thus we need (i) and (ii). The other cases are similar. □

**Lemma 4**

Every rudimentary function is $\Sigma_0$ in the parameters $X_i$.

**Proof:** By induction, one proves that the rudimentary functions that are generated without the schema $f(x_1,\ldots,x_n) = X_i \cap x_j$ are simple. For this, one uses lemma 3. But since the function $f(x,y) = x \cap y$ is one of those, the claim holds. □

Thus every rudimentary relation is $\Sigma_0$ in the parameters $X_i$, but not necessarily $\Sigma_0$ with the $X_i$ as predicates. An example is the relation $\{x,y\} \in X_0$.

A structure is said to be rudimentary closed if its underlying set is closed under all rudimentary functions.

**Lemma 5**

If $\mathfrak{M}$ is rudimentary closed and $H \prec_1 \mathfrak{M}$, then $H$ and the collapse of $H$ are also rudimentary closed.

**Proof:** That is clear, since the functions $F_0,\ldots,F_{9+i}$ are $\Sigma_0$ with the predicates $X_i$. □

Let $T_N$ be the set of $\Sigma_0$ formulae of our language $\{\in,X_1,\ldots,X_N\}$ having exactly one free variable. By lemma 2, there is a rudimentary function $f$ for every $\Sigma_0$ formula $\psi$ such that $\psi(x) \Leftrightarrow f(x) \neq \emptyset$. By lemma 1, we have

\[ x_0 = f(x_*) = F_{k_1}(x_1,x_2) \]
\[ x_1 = F_{k_2}(x_3,x_4) \]
\[ x_2 = F_{k_3}(x_5,x_6) \]
\[ x_3 = \ldots \]
Of course, $x_*$ appears at some point.
Therefore, we may define an effective Gödel coding

$$ T_N \to G, \psi_u \mapsto u $$

as follows ($m, n$ possibly $= *$):

$$ (k, l, m, n) \in u \iff x_k = F_l(x_m, x_n). $$

Let $\models_{\mathfrak{M}} (u, x_*) \iff$

- $\psi_u$ is a $\Sigma_0$ formula with exactly one free variable
- and $\mathfrak{M} \models \psi_u(x_*)$.

**Lemma 6**

If $\mathfrak{M}$ is transitive and rudimentary closed, then $\models_{\mathfrak{M}} (x, y)$ is $\Sigma_1$-definable over $\mathfrak{M}$. The definition of $\models_{\mathfrak{M}} (x, y)$ depends only on the number of predicates of $\mathfrak{M}$. That is, it is uniform for all structures of the same type.

**Proof:** Whether $\models_{\mathfrak{M}} (u, x_*)$ holds, may be computed directly. First, one computes the $x_k$ which only depend on $x_*$. For those $k$, $(k, l, *, *) \in u$. Then one computes the $x_i$ which only depend on $x_m$ and $x_n$ such that $m, n \in \{ k \mid (k, l, m, n) \in u \}$ etc. Since $\mathfrak{M}$ is rudimentary closed, this process only breaks off, when one has computed $x_0 = f(x_*)$. And $\models_{\mathfrak{M}} (u, x_*)$ holds iff $x_0 = f(x_*) \neq \emptyset$.

More formally speaking: $\models_{\mathfrak{M}} (u, x_*)$ holds iff there is some sequence $(x_i \mid i \in d)$, $d = \{ k \mid (k, l, m, n) \in u \}$ such that

- $(k, l, m, n) \in u \Rightarrow x_k = F_l(x_m, x_n)$
- and $x_0 \neq \emptyset$.

Hence $\models_{\mathfrak{M}}$ is $\Sigma_1$. \qed

If $\mathfrak{M}$ is a structure, then let $\text{rud}(\mathfrak{M})$ be the closure of $W \cup \{ W \}$ under the functions which are rudimentary for $\mathfrak{M}$.

**Lemma 7**

If $\mathfrak{M}$ is transitive, then so is $\text{rud}(\mathfrak{M})$.

**Proof:** By induction on the definition of the rudimentary functions. \qed

**Lemma 8**

Let $\mathfrak{M}$ be a transitive structure with underlying set $W$. Then

$$ \text{rud}(\mathfrak{M}) \cap \mathcal{P}(W) = \text{Def}(\mathfrak{M}). $$

**Proof:** First, let $A \in \text{Def}(\mathfrak{M})$. Then $A$ is $\Sigma_0$ over $\{ W \cup \{ W \}, X_i \}$, i.e. there are parameters $p_i \in W \cup \{ W \}$ and some $\Sigma_0$ formula $\varphi$ such that $x \in A \iff \varphi(x, p_i)$. But by lemma 2, every $\Sigma_0$ relation is rudimentary. Thus there is a rudimentary function $f$ such that $x \in A \iff f(x, p_i) \neq \emptyset$. Let $g(z, x) = \{ x \}$ and define $h(y, x) = \bigcup \{ g(z, x) \mid z \in y \}$. Then $h(f(x, p_i), x) = \bigcup \{ g(z, x) \mid z \in f(x, p_i) \}$ is rudimentary, $h(f(x, p_i), x) = \emptyset$ if $x \notin A$ and $h(f(x, p_i), x) = \{ x \}$ if $x \in A$. Finally, let $H(y, p_i) = \bigcup \{ h(f(x, p_i), x) \mid x \in y \}$. Then $H$ is rudimentary and $A = H(W, p_i)$. So we are done.
Conversely, let $A \in rud(\mathfrak{M}) \cap \mathfrak{P}(W)$. Then there is a rudimentary function $f$ and some $a \in W$ such that $A = f(a, W)$. By lemma 4 and lemma 3, there exists a $\Sigma_0$ formula $\psi$ such that $x \in f(a, W) \iff \psi(x, a, W, X_i)$. By $\Sigma_0$ absoluteness, $A = \{x \in W \mid W \cup \{W, X_i\} \models \psi(x, a, W, X_i)\}$, since $X_i \subseteq W$. Therefore, there is a formula $\varphi$ such that $A = \{x \in W \mid \mathfrak{M} \models \varphi(x, a)\}$. □

Let $\kappa \in \text{Card} - \omega_1$, $S^X \subseteq \text{Lim} \cap \kappa$ and $\langle X_\nu \mid \nu \in S^X \rangle$ be a sequence.

For $\nu \in \text{Lim} - S^X$, let $I_\nu = \langle J^X_\nu, X \mid \nu \rangle$ and let $I_\nu = \langle J^X_\nu, X \mid \nu, X_\nu \rangle$ for $\nu \in S^X$ such that $X_\nu \subseteq J^X_\nu$ where

$$J^X_0 = \emptyset$$

$$J^X_\nu+\omega = rud(I_\nu)$$

$$J^X_\nu = \bigcup \{J^X_\nu \mid \nu \in \lambda\} \text{ if } \lambda \in \text{Lim}^2 := \text{Limit}.$$

Obviously, $I_\nu[X] = \bigcup \{J^X_\nu \mid \nu \in \kappa\}$.

We say that $L_\kappa[X]$ is amenable if $I_\nu$ is rudimentary closed for all $\nu \in S^X$.

**Lemma 9**

(i) Every $J^X_\nu$ is transitive

(ii) $\mu < \nu \Rightarrow J^X_\mu \subseteq J^X_\nu$

(iii) $\text{rank}(J^X_\nu) = J^X_\nu \cap \text{On} = \nu$

**Proof:** That are three easy proofs by induction. □

Sometimes we need levels between $J^X_\nu$ and $J^X_{\nu+\omega}$. To make those transitive, we define

$$G_1(x, y, z) = F_i(x, y) \text{ for } i \leq 8$$

$$G_9(x, y, z) = x \cap X$$

$$G_{10}(x, y, z) = \langle x, y \rangle$$

$$G_{11}(x, y, z) = \langle x, y \rangle$$

$$G_{12}(x, y, z) = \{x, y\}$$

$$G_{13}(x, y, z) = \langle x, y, z \rangle$$

$$G_{14}(x, y, z) = \{x, y, z\}.$$  

Let

$$S_0 = \emptyset$$

$$S_{\mu+1} = S_\mu \cup \{S_\mu \cup \bigcup \{G_i([S_\mu \cup \{S_\mu\}]^3) \mid i \leq 15\}\}$$

$$S_\lambda = \bigcup \{S_\mu \mid \mu \in \lambda\} \text{ if } \lambda \in \text{Lim}.$$  

**Lemma 10**

The sequence $\langle I_\mu \mid \mu \in \text{Lim} \cap \nu \rangle$ is (uniformly) $\Sigma_1$-definable over $I_\nu$.  

**Proof:** By definition $J^X_\mu = S_\mu$ for $\mu \in \text{Lim}$, that is, the sequence $\langle J^X_\mu \mid \mu \in \text{Lim} \cap \nu \rangle$ is the solution of the recursion defining $S_\mu$ restricted to $\text{Lim}$. Since the recursion condition is $\Sigma_0$ over $I_\nu$, the solution is $\Sigma_1$. It is $\Sigma_1$ over $I_\nu$ if the existential quantifier can be restricted to $J^X_\nu$. Hence we must prove $\langle S_\mu \mid \mu \in \tau \rangle \in J^X_\nu$ for $\tau \in \nu$. This is done by induction on $\nu$. The base case $\nu = 0$ and the limit step are clear. For the successor step, note that $S_{\mu + 1}$ is a rudimentary function of $S_\mu$ and $\mu$, and use the rudimentary closedness of $J^X_\nu$.  

6
Lemma 11
There are well-orderings $<_{\nu}$ of the sets $J^X_{\nu}$ such that

(i) $\mu < \nu \Rightarrow <_{\mu} \subseteq <_{\nu}$

(ii) $<_{\nu+1}$ is an end-extension of $<_{\nu}$

(iii) The sequence $\langle <_{\mu} \mid \mu \in \text{Lim} \cap \nu \rangle$ is (uniformly) $\Sigma_1$-definable over $I_{\nu}$.

(iv) $<_{\nu}$ is (uniformly) $\Sigma_1$-definable over $I_{\nu}$.

(v) The function $pr_{\nu}(x) = \{ z \mid z <_{\nu} x \}$ is (uniformly) $\Sigma_1$-definable over $I_{\nu}$.

Proof: Define well-orderings $<_{\mu}$ of $S_{\mu}$ by recursion:

(I) $<_{0} = \emptyset$

(II) (1) For $x, y \in S_{\mu}$, let $x <_{\mu+1} y \iff x <_{\mu} y$

(2) $x \in S_{\mu}$ and $y \notin S_{\mu} \Rightarrow x <_{\mu+1} y$

(3) If $x, y \notin S_{\mu}$, then there is an $i \in 15$ and $x_1, x_2, x_3 \in S_{\mu}$ such that $x = G_i(x_1, x_2, x_3)$ and there is a $j \in 15$ and $y_1, y_2, y_3 \in S_{\mu}$ such that $y = G_j(y_1, y_2, y_3)$. First, choose $i$ and $j$ minimal, then $x_1$ and $y_1$, then $x_2$ and $y_2$, and finally $x_3$ and $y_3$.

Set:

(a) $x <_{\mu+1} y$ if $i < j$

(b) $x <_{\mu+1} y$ if $i = j$ and $x_1 <_{\mu} y_1$

(c) $x <_{\mu+1} y$ if $i = j$ and $x_1 = y_1$ and $x_2 <_{\mu} y_2$

(d) $x <_{\mu+1} y$ if $i = j$ and $x_1 = y_1$ and $x_2 = y_2$ and $x_3 <_{\mu} y_3$

(III) $<_{\lambda} = \bigcup \{ <_{\mu} \mid \mu \in \lambda \}$

The properties (i) to (v) are obvious. For the $\Sigma_1$-definability, one needs the argument from lemma 10. $\square$

Lemma 12
The rudimentary closed $\langle J^X_{\nu}, X \upharpoonright \nu, A \rangle$ have a canonical $\Sigma_1$-Skolem function $h$.

Proof: Let $\langle \psi_i \mid i \in \omega \rangle$ be an effective enumeration of the $\Sigma_0$ formulae with three free variables. Intuitively, we would define:

$$h(i, x) \simeq (z)_0$$

for

the $<_{\nu}$-least $z \in J^X_{\nu}$ such that $\langle J^X_{\nu}, X \upharpoonright \nu, A \rangle \models \psi_i((z)_0, x, (z)_1)$.

Formally, we define:
By lemma 11 (v), let \( \theta \) be a \( \Sigma_0 \) formula such that
\[
w = \{ v \mid v < \nu \ z \} \iff \langle J^X_\nu, X \mid \nu, A \rangle \models (\exists t) \theta(w, z, t).
\]
Let \( u_i \) be the Gödel coding of
\[
\theta((s)_1, (s)_0, (s)_2)
\]
\[
\land \psi_i(((s)_0)_0, (s)_3, ((s)_0)_1) \land (\forall v \in (s)_1) \neg \psi_i((v)_0, (s)_3, (v)_1)
\]
and
\[
y = h(i, x) \iff
(\exists s)(((s)_0)_0 = y \land (s)_3 = x \land \models_{(J^X_\nu, X \mid [\nu, A]}} (u_i, s)).
\]
This has the desired properties. Note lemma 6! \( \square \)

I will denote this \( \Sigma_1 \)-Skolem function by \( h_{\nu, A} \). Let \( h_{\nu, 0} := h_{\nu, A} \).

Let us say that \( L_\kappa [X] \) has condensation if the following holds:
If \( \nu \in S^X \) and \( H \prec_1 L_\kappa \), then there is some \( \mu \in S^X \) such that \( H \cong L_\mu \).

From now on, suppose that \( L_\kappa [X] \) is amenable and has condensation.

Set \( I^\nu_0 = \langle J^X_\nu, X \mid \nu \rangle \) for all \( \nu \in \operatorname{Lim} \cap \kappa \).

**Lemma 13** (Gödel’s pairing function)
There is a bijection \( \Phi : On^2 \to On \) such that \( \Phi(\alpha, \beta) \geq \alpha, \beta \) for all \( \alpha, \beta \) and \( \Phi^{-1} \mid \alpha \) is uniformly \( \Sigma_1 \)-definable over \( I^\alpha_0 \) for all \( \alpha \in \operatorname{Lim} \).

**Proof:** Define a well-ordering \( \prec^* \) on \( On^2 \) by
\[
(\alpha, \beta) \prec^* (\gamma, \delta)
\]
iff
\[
\begin{align*}
\max(\alpha, \beta) &< \max(\gamma, \delta) \\
\max(\alpha, \beta) &= \max(\gamma, \delta) \text{ and } \alpha < \gamma \text{ or } \\
\max(\alpha, \beta) &= \max(\gamma, \delta) \text{ and } \alpha = \gamma \text{ and } \beta < \delta.
\end{align*}
\]
Let \( \Phi : (On^2, \prec^*) \cong (On, \prec) \). Then \( \Phi \) may be defined by the recursion
\[
\Phi(0, \beta) = \sup(\Phi(\nu, \nu) \mid \nu < \beta)
\]
\[
\Phi(\alpha, \beta) = \Phi(0, \beta) + \alpha \text{ if } \alpha < \beta.
\]
\[
\Phi(\alpha, \beta) = \Phi(0, \alpha) + \alpha + \beta \text{ if } \alpha \geq \beta.
\]
\( \square \)

So there is a uniform map from \( \alpha \) onto \( \alpha \times \alpha \) for all \( \alpha \) that are closed under Gödel’s pairing function. Such a map exists for all \( \alpha \in \operatorname{Lim} \). But then we have to give up uniformity.

**Lemma 14**
For all \( \alpha \in \operatorname{Lim} \), there exists a function from \( \alpha \) onto \( \alpha \times \alpha \) that is \( \Sigma_1 \)-definable over \( I^\alpha_0 \).

**Proof** by induction on \( \alpha \in \operatorname{Lim} \). If \( \alpha \) is closed under Gödel’s pairing function, then lemma 13 does the job. Therefore, if \( \alpha = \beta + \omega \) for some \( \beta \in \operatorname{Lim} \), we may assume \( \beta \neq 0 \). But then there is some over \( I^\beta_0 \) \( \Sigma_1 \)-definable bijection \( j : \alpha \to \beta \). And by the induction hypothesis, there is an over \( I^\beta_0 \) \( \Sigma_1 \)-definable function from \( \beta \) onto \( \beta \times \beta \). Thus there exists a \( \Sigma_1 \) formula \( \varphi(x, y, p) \) and a parameter \( p \in J^X_\beta \) such that there is some \( x \in \beta \) satisfying \( \varphi(x, y, p) \) for all \( y \in \beta \times \beta \). So we
get an over $P_0^0\Sigma_1$-definable injective function $g : \beta \times \beta \to \beta$ from the $\Sigma_1$-Skolem function. Hence $f((\nu, \tau)) = g(j(\nu), j(\tau)))$ defines an injective function $f : \alpha^2 \to \beta$ which is $\Sigma_1$-definable over $P_0^0\alpha$. An $h$ which is as needed may be defined by

$$h(\nu) = f^{-1}(\nu) \text{ if } \nu \in \text{rng}(f)$$

$$h(\nu) = (0,0) \text{ else.}$$

For $\text{rng}(f) = \text{rng}(g) \in J^{\Sigma_1}_0$. 

Now, assume $\alpha \in \text{Lim}^2$ is not closed under Gödel’s pairing function. Then $\nu, \tau \in \alpha$ for $\langle \nu, \tau \rangle = \Phi^{-1}(\alpha)$, and $e := \{z \mid z <^* \langle \nu, \tau \rangle \}$ lies in $J^{\Sigma_1}_\alpha$. Thus $\Phi^{-1} : e \to \alpha$ is an over $P_0^0\Sigma_1$-definable bijection. Pick a $\gamma \in \text{Lim}$ such that $\nu, \tau < \gamma$. Then $\Phi^{-1} \upharpoonright \alpha : \alpha \to \gamma^2$ is an over $P_0^0\Sigma_1$-definable injective function. Like in the first case, there exists an injective function $g : \gamma \times \gamma \to \gamma$ in $J^{\Sigma_1}_\alpha$ by the induction hypothesis. So $f(\langle \xi, \zeta \rangle) = g(g(\Phi^{-1}(\xi), g(\Phi^{-1}(\zeta))))$ defines an over $P_0^0\Sigma_1$-definable bijection $f : \alpha^2 \to d$ such that $d := g[g[c] \times g[c]]$. Again, we define $h$ by

$$h(\xi) = f^{-1}(\xi) \text{ if } \xi \in d$$

$$h(\xi) = (0,0) \text{ else.} \quad \Box$$

**Lemma 15**

Let $\alpha \in \text{Lim} - \omega + 1$. Then there is some over $P_0^0\Sigma_1$-definable function from $\alpha$ onto $J^{\Sigma_1}_\alpha$. This function is uniformly definable for all $\alpha$ closed under Gödel’s pairing function.

**Proof:** Let $f : \alpha \to \alpha \times \alpha$ be a surjective function which is $\Sigma_1$-definable over $P_0^0\alpha$ with parameter $p$. Let $\rho$ be minimal with respect to the canonical well-ordering such that such an $f$ exists. Define $f_0^0, f_1^1$ by $f(\nu) = (f_0^0(\nu), f_1^1(\nu))$ and, by induction, define $f_1 = \text{id} \upharpoonright \alpha$ and $f_{n+1}(\nu) = (f_0^0(\nu), f_n \circ f_1^1(\nu))$. Let $h := h_\alpha$ be the canonical $\Sigma_1$-Skolem function and $H = h(\omega \times (\alpha \times \{\rho\}))$. Then $H$ is closed under ordered pairs. For, if $y_1 = h(j_1, \langle \nu_1, p \rangle)$, $y_2 = h(j_2, \langle \nu_2, p \rangle)$ and $\langle \nu_1, \nu_2 \rangle = f(\tau)$, then $\langle y_1, y_2 \rangle$ is $\Sigma_1$-definable over $P_0^0\alpha$ with the parameters $\tau, p$. Hence it is in $H$. Since $H$ is closed under ordered pairs, we have $H \sim_{P_0^0\alpha} H$. Let $\sigma : H \to P_0^0\alpha$ be the collapse of $H$. Then $\alpha = \beta$, because $\alpha \subseteq H$ and $\sigma \upharpoonright \alpha = \text{id} \upharpoonright \alpha$. Thus $\sigma[f] = f_1$ and $\sigma[f]$ is $\Sigma_1$-definable over $P_0^0\alpha$ with the parameter $\sigma(p)$. Since $\sigma$ is a collapse, $\sigma(p) \leq p$. So $\sigma(p) = p$ by the minimality of $p$. In general, $\pi(h(i, x)) \cong h(i, \pi(x))$ for $\Sigma_1$-elementary $\pi$. Therefore, $\sigma(h(i, \langle \nu, p \rangle)) \cong h(i, \langle \nu, p \rangle)$ holds in our case for all $\nu \in \alpha$. But then $\sigma \upharpoonright H = \text{id} \upharpoonright \alpha$ and $H = J^{\Sigma_1}_\alpha$. Thus we may define the needed surjective map by $g \circ f_3$ where

$$g(i, \nu, \tau) = y \text{ if } (\exists \nu \in S_\tau) \varphi(z, y, i, \langle \nu, p \rangle)$$

$$g(i, \nu, \tau) = \emptyset \text{ else.}$$

Here, $S_\tau$ shall be defined as in lemma 10 and $y = h(i, x) \Leftrightarrow (\exists t \in J^{\Sigma_1}_\alpha) \varphi(t, i, x, y)$. \Box

Let $(P_0^0, A) := (J^{\Sigma_1}_\nu, X \upharpoonright \nu, A)$. The idea of the fine structure theory is to code $\Sigma_\alpha$ predicates over large structures in $\Sigma_1$ predicates over smaller structures. In the simplest case, one codes the $\Sigma_1$ information of the given structure $P_0^0$ in a rudimentary closed structure $(P_0^0, A)$. I.e. we want to have something like:

Over $P_0^0\beta$, there exists a $\Sigma_1$ function $f$ such that

$$f[J^{\Sigma_1}_\beta] = J^{\Sigma_1}_\beta.$$
For the \( \Sigma_1 \) formulae \( \varphi_i \),

\[
\langle i, x \rangle \in A \iff I_0^\beta \models \varphi_i(f(x))
\]

holds. And

\[
(I_0^\beta, A) \text{ is rudimentary closed.}
\]

Now, suppose we have such an \( (I_0^\beta, A) \). Then every \( B \subseteq J_0^X \) that is \( \Sigma_1 \)-definable over \( I_0^\beta \) is of the form

\[
B = \{ x \mid A(i, \langle x, p \rangle) \} \quad \text{for some} \quad i \in \omega, p \in J_0^X.
\]

So \( (I_0^\beta, B) \) is rudimentary closed for all \( B \in \Sigma_1(I_0^\beta) \cap \mathcal{P}(J_0^X) \).

The \( \rho \) is uniquely determined.

**Lemma 16**

Let \( \beta > \omega \) and \( (I_0^\beta, B) \) be rudimentary closed. Then there is at most one \( \rho \in \text{Lim} \) such that

\[
(I_0^\beta, C) \text{ is rudimentary closed for all } C \in \Sigma_1((I_0^\beta, B)) \cap \mathcal{P}(J_0^X)
\]

and

there is an over \( (I_0^\beta, B) \) \( \Sigma_1 \)-definable function \( f \) such that \( f[J_0^X] = J_0^X \).

**Proof:** Assume \( \rho < \tilde{\rho} \) both had these properties. Let \( f \) be an over \( (I_0^\beta, B) \) \( \Sigma_1 \)-definable function such that \( f[J_0^X] = J_0^X \) and \( C = \{ x \in J_0^X \mid x \not\in f(x) \} \).

Then \( C \subseteq J_0^X \) is \( \Sigma_1 \)-definable over \( (I_0^\beta, B) \). So \( (I_0^\beta, C) \) is rudimentary closed.

But then \( C \cap J_0^X \in J_0^X \). Hence there is an \( x \in J_0^X \) such that \( C = f(x) \).

From this, the contradiction \( x \in f(x) \iff x \in C \iff x \not\in f(x) \) follows. \( \square \)

The uniquely determined \( \rho \) from lemma 16 is called the projectum of \( (I_0^\beta, B) \).

If there is some over \( (I_0^\beta, B) \) \( \Sigma_1 \)-definable function \( f \) such that \( f[J_0^X] = J_0^X \),

then \( h_{\beta, B} \omega \times (J_0^X \times \{ p \}) = J_0^X \) for a \( p \in J_0^X \). Using the canonical function \( h_{\beta, B} \), we can define a canonical \( A \):

Let \( p \) be minimal with respect to the canonical well-ordering such that the above property holds. Define

\[
A = \{ \langle i, x \rangle \mid i \in \omega \text{ and } x \in J_0^X \text{ and } (I_0^\beta, B) \models \varphi_i(x, p) \}.
\]

We say \( p \) is the standard parameter of \( (I_0^\beta, B) \) and \( A \) the standard code of it.

**Lemma 17**

Let \( \beta > 0 \) and \( (I_0^\beta, B) \) be rudimentary closed. Let \( \rho \) be the projectum and \( A \) the standard code of it. Then for all \( m \geq 1 \), the following holds:

\[
\Sigma_{1+m}(\langle I_0^\beta, B \rangle) \cap \mathcal{P}(J_0^X) = \Sigma_m(\langle I_0^\beta, A \rangle).
\]

**Proof:** First, let \( R \in \Sigma_{1+m}(\langle I_0^\beta, B \rangle) \cap \mathcal{P}(J_0^X) \) and let \( m \) be even. Let \( P \) be a relation being \( \Sigma_1 \)-definable over \( \langle I_0^\beta, B \rangle \) with parameter \( q_1 \) such that, for \( x \in J_0^X \), \( R(x) \) holds if \( \exists y_0 \forall y_1 \exists y_3 \ldots \forall y_{m-1} P(y_1, x) \). Let \( f \) be some over \( (I_0^\beta, B) \) with parameter \( q_2 \) \( \Sigma_1 \)-definable function such that \( f[J_0^X] = J_0^X \). Define \( Q(z_i, x) \) by \( z_i, x \in J_0^X \) and \( (\exists y_i)(y_i = f(z_i) \text{ and } P(y_i, x)) \). Let \( p \) be the standard parameter
of \((I^0_\beta, B)\). Then, by definition, there is some \(u \in J^X_\rho\) such that \((q_1, q_2)\) is \(\Sigma_1\)-definable in \((I^0_\beta, B)\) with the parameters \(u, p\). I.e. there is some \(i \in \omega\) such that \(Q(z_i, x)\) holds iff \(z_i, x \in J^X_\rho\) and \((I^0_\beta, B) \models \varphi_i((z_i, x, u), p)\) i.e. iff \(z_i, x \in J^X_\rho\) and \(A(i, (z_i, x, u))\). Analogously there is a \(j \in \omega\) and a \(v \in J^X_\rho\) such that \(z \in \text{dom}(f) \land J^X_\rho\) iff \(z \in J^X_\rho\) and \(A(j, (z, v))\). Abbreviate this by \(D(z)\). But then, for \(x \in J^X_\rho\), \(R(x)\) holds iff \(\exists y \forall z_1 \exists y_3 \ldots \exists y_{m-1} (D(z_0) \land D(z_2) \land \ldots \land D(z_{m-2})\) and \((D(z_1) \land D(z_3) \land \ldots \land D(z_{m-1}) \Rightarrow Q(z_i, x))\). So the claim holds. If \(m\) is odd, then we proceed correspondingly. Thus \(\Sigma_{1+m}(\langle I^0_\beta, B \rangle) \cap \Phi(J^X) \subseteq \Sigma_m(\langle I^0_\beta, A \rangle)\) is proved.

Conversely, let \(\varphi\) be a \(\Sigma_0\) formula and \(q \in J^X_\rho\) such that, for all \(x \in J^X_\rho\), \(R(x)\) holds iff \(\langle I^0_\beta, A \rangle \models \varphi(x, q)\). Since \(\langle I^0_\beta, A \rangle\) is rudimentary closed, \(R(x)\) holds iff \((\exists u \in J^X_\rho)(\exists v \in J^X_\rho)(u \text{ transitive and } x \in u\) and \(q \in u\) and \(a = A \cup u\) and \((u, a) \models \varphi(x, q)\). Write \(a = A \cup u\) as formula: \((\forall v \in a)(v \in u \land v \in A)\) and \((\forall v \in u)(v \in A \Rightarrow v \in a)\). If \(m = 1\), we are done provided we can show that this is \(\Sigma_2\) over \(\langle I^0_\beta, B \rangle\). If \(m > 1\), the claim follows immediately by induction. The second part is \(\Pi_1\). So we only have to prove that the first part is \(\Sigma_2\) over \(\langle I^0_\beta, B \rangle\).

By the definition of \(A, v \in A\) is \(\Sigma_1\)-definable over \(\langle I^0_\beta, B \rangle\). I.e. there is some \(\Sigma_0\) formula \(\psi\) and some parameter \(p\) such that \(v \in A \Leftrightarrow (\langle I^0_\beta, B \rangle \models (\exists y)\psi(v, y, p)\).

Now, we have two cases.

In the first case, there is no over \(\langle I^0_\beta, B \rangle\) \(\Sigma_1\)-definable function from some \(\gamma < \rho\) cofinal in \(\beta\). Then \((\forall v \in a)(v \in A)\) is \(\Sigma_2\) over \(\langle I^0_\beta, B \rangle\), because some kind of replacement axiom holds, and \(\forall (\forall v \in a)(\exists y)\psi(v, y, p)\) is over \(\langle I^0_\beta, B \rangle\) equivalent to \((\exists z)(\forall v \in a)(\exists y \in z)\psi(v, y, p)\). For \(\rho = \omega\), this is obvious. If \(\rho \neq \omega\), then \(\rho \in \text{Lim}'\) and we can pick a \(\gamma < \rho\) such that \(a \in J^X_\rho\). Let \(j : \gamma \rightarrow J^X_\rho\) an over \(I, \Sigma_1\)-definable surjection, and \(g\) an over \(\langle I^0_\beta, B \rangle\) \(\Sigma_1\)-definable function that maps \(v \in J^X_\rho\) to \(g(v) \in J^X_\rho\) such that \(\psi(v, g(v), p)\) if such an element exists. We can find such a function with the help of the \(\Sigma_1\)-Skolem function. Now, define a function \(f : \gamma \rightarrow \beta\) by

\[
 f(\nu) = \text{the least } \tau < \beta \text{ such that } g \circ j(\nu) \in S_\tau \text{ if } j(\nu) \in a \\
 f(\nu) = 0 \text{ else.}
\]

Since \(f\) is \(\Sigma_1\), there is, in the given case, a \(\delta < \beta\) such that \(f[\gamma] \subseteq \delta\). So we have as collecting set \(z = S_\delta\), and the equivalence is clear.

Now, let us come to the second case. Let \(\gamma < \rho\) be minimal such that there is some over \(\langle I^0_\beta, B \rangle\) \(\Sigma_1\)-definable function \(q\) from \(\gamma\) cofinal in \(\beta\). Then \((\forall v \in a)(\exists y)\psi(v, y, p)\) is equivalent to \((\forall v \in a)(\exists \nu \in \gamma)(\exists y \in S_{g(\nu)}\psi)(v, y, p)\). If we define a predicate \(C \subseteq J^X_\rho\) by \((v, \nu) \in C \iff y \in S_{g(\nu)}\) and \(\psi(v, y, p)\), then \((I^0_\beta, B) \models (\forall v \in a)(\exists \nu \in \gamma)(\exists y \in C) \iff (\forall v \in a)(\exists \nu \in \gamma)(\exists y \in C)\). But this holds iff \(\langle I^0_\beta, C \rangle \models (\exists w)(w \text{ transitive and } \gamma \in w \text{ and } (w, C \cap w) \models (\forall v \in a)(\exists \nu \in \gamma)(\exists y \in C)\). Since \(C\) is \(\Sigma_1\) over \(\langle I^0_\beta, B \rangle\), \(\langle I^0_\beta, C \rangle\) is rudimentary closed by the definition of the projectum. I.e. the statement is equivalent to \(\langle I^0_\beta, C \rangle \models (\exists w)(\exists c)(w \text{ transitive and } \gamma \in w \text{ and } c = C \cap w \text{ and } (w, c) \models (\forall v \in a)(\exists \nu \in \gamma)(\exists y \in c)\). So, to prove that this is \(\Sigma_2\), it suffices to show that \(c = C \cap w\) is \(\Sigma_2\). In its full form, this is \((\forall z)(z \in a \Leftrightarrow z \in w \text{ and } z \in C)\). But \(z \in C\) is even \(\Delta_1\) over \(\langle I^0_\beta, B \rangle\) by the definition. So we are finished. \(\square\)

**Lemma 18**

(a) Let \(\pi : (J^X_\beta, X \upharpoonright \beta, B) \rightarrow (J^X_\beta, X \upharpoonright \beta, B)\) be \(\Sigma_0\)-elementary and \(\pi[\beta]\) be
cofinal in $\beta$. Then $\pi$ is even $\Sigma_1$-elementary.

(b) Let $\langle J^X_\nu, X \mid \nu, A \rangle$ be rudimentary closed and $\pi : \langle J^X_\nu, X \mid \nu \rangle \rightarrow \langle J^Y_\nu, Y \mid \nu \rangle$ be $\Sigma_0$-elementary and cofinal. Then there is a uniquely determined $A \subseteq J^X_\nu$ such that $\pi : \langle J^X_\nu, X \mid \nu, A \rangle \rightarrow \langle J^X_\nu, X \mid \nu, A \rangle$ is $\Sigma_0$-elementary and $\langle J^X_\nu, X \mid \nu, A \rangle$ is rudimentary closed.

**Proof:** (a) Let $\varphi$ be a $\Sigma_0$ formula such that $\langle J^X_\nu, X \mid \beta, B \rangle \models (\exists z) \varphi(z, \pi(x_i))$. Since $\pi[\bar{\beta}]$ is cofinal in $\beta$, there is a $\nu \in \beta$ such that $\langle J^X_\nu, X \mid \beta, B \rangle \models (\exists z \in S_{\pi(\nu)}) \varphi(z, \pi(x_i))$. Here, the $S_\nu$ is defined as in lemma 10. If $\pi(S_\nu) = S_{\pi(\nu)}$, then $\langle J^X_\nu, X \mid \beta, B \rangle \models (\exists z \in \pi(S_\nu)) \varphi(z, \pi(x_i))$. So, by the $\Sigma_0$-elementarity of $\pi$, $\langle J^X_\nu, X \mid \beta, B \rangle \models (\exists z \in S_\nu) \varphi(z, x_i)$. I.e. $\langle J^X_\nu, X \mid \beta, B \rangle \models (\exists z) \varphi(z, x_i)$. The converse is trivial.

It remains to prove $\pi(S_\nu) = S_{\pi(\nu)}$. This is done by induction on $\nu$. If $\nu = 0$ or $\nu \notin Lim$, then the claim is obvious by the definition of $S_\nu$ and the induction hypothesis. So let $\lambda \in Lim$ and $M := \pi(S_\lambda)$. Then $M$ is transitive by the $\Sigma_0$-elementarity of $\pi$. And since $\lambda \in Lim$ (i.e. $S_\lambda = J^X_0$), $(S_\nu \mid \nu < \lambda)$ is definable over $(J^X_\lambda, X \mid \lambda)$ (by the proof of) lemma 10. Let $\varphi$ be the formula $\forall x (\exists y)(x \in S_\nu)$. Since $\pi$ is $\Sigma_0$-elementary, $\pi \upharpoonright S_\lambda : \langle J^X_\lambda, X \mid \lambda \rangle \rightarrow (M, (X \mid \lambda) \cap M)$ is elementary. Thus, if $\langle J^X_\lambda, X \mid \lambda \rangle \models \varphi$, then $\langle M, (X \mid \lambda) \cap M \rangle \models \varphi$. Since $M$ is transitive, we get $M = S_\nu$ for a $\tau \in Lim$. And, by $\pi(\lambda) = \pi(S_\lambda \cap On) = S_\nu \cap On = \tau$, it follows that $\pi(S_\lambda) = S_{\pi(\lambda)}$.

(b) Since $\langle J^X_\nu, X \mid \nu, A \rangle$ is rudimentary closed, $A \cap S_\mu \in J^X_{\nu}$ for all $\mu < \nu$ where $S_\mu$ is defined as in lemma 10. As in the proof of (a), $\pi(S_\mu) = S_{\pi(\mu)}$. So we need $\pi(A \cap S_\mu) = A \cap S_{\pi(\mu)}$ to get that $\pi : \langle J^X_\nu, X \mid \nu, A \rangle \rightarrow \langle J^Y_\nu, Y \mid \nu, A \rangle$ is $\Sigma_0$-elementary. Since $\pi$ is cofinal, we necessarily obtain $A = \bigcup \{ \pi(A \cap S_\mu) \mid \mu < \nu \}$. But then $\langle J^Y_\nu, X \mid \nu, A \rangle$ is rudimentary closed. For, if $x \in J^X_{\nu}$, we can choose some $\mu < \nu$ such that $x \in S_{\pi(\mu)}$. And $x \cap A = x \cap (A \cap S_{\pi(\mu)}) = x \cap \pi(A \cap S_{\pi(\mu)}) \in J^X_{\nu}$. Now, let $\langle J^X_\nu, X \mid \nu, A \rangle \models \varphi(x_i)$ where $\varphi$ is a $\Sigma_0$ formula and $u \in J^X_{\nu}$ is transitive such that $x_i \in u$. Then $\langle u, X \mid \nu \cap u, A \cap u \rangle \models \varphi(x_i)$ holds. Since $\pi : \langle J^X_\nu, X \mid \nu \rangle \rightarrow \langle J^Y_\nu, Y \mid \nu \rangle$ is $\Sigma_0$-elementary, $(\pi(u), Y \mid \nu \cap \pi(u), A \cap \pi(u)) \models \varphi(\pi(x_i))$. Because $\pi(u)$ is transitive, we get $\langle J^Y_\nu, X \mid \nu, A \rangle \models \varphi(\pi(x_i))$. This argument works as well for the converse. \qed

Write $Cond_B(I^0_\beta)$ if there exists for all $H \triangleleft_1 \langle I^0_\beta, B \rangle$ some $\beta$ and some $B$ such that $H \equiv \langle I^0_\beta, B \rangle$.

**Lemma 19** (Extension of embeddings)

Let $\beta > \omega$, $m \geq 0$ and $\langle I^0_\beta, B \rangle$ be a rudimentary closed structure. Let $Cond_B(I^0_\beta)$ hold. Let $\rho$ be the projectum of $\langle I^0_\beta, B \rangle$, $A$ the standard code and $p$ the standard parameter of $\langle I^0_\beta, B \rangle$. Then $Cond_A(I^0_\beta)$ holds. And if $\langle I^0_\beta, A \rangle$ is rudimentary closed and $\pi : \langle I^0_\beta, A \rangle \rightarrow \langle I^0_\beta, A \rangle$ is $\Sigma_m$-elementary, then there is an uniquely determined $\Sigma_{m+1}$-elementary extension $\tilde{\pi} : \langle I^0_\beta, \tilde{B} \rangle \rightarrow \langle I^0_\beta, B \rangle$ of $\pi$ where $\tilde{\rho}$ is the projectum of $\langle I^0_\beta, \tilde{B} \rangle$, $\tilde{A}$ is the standard code and $\tilde{\pi}^{-1}(p)$ is the standard parameter of $\langle I^0_\beta, B \rangle$.

**Proof:** Let $H = h_{\beta, B}[\omega \times (\text{rng}(\pi) \times \{p\})] \triangleleft_1 \langle I^0_\beta, B \rangle$ and $\tilde{\pi} : \langle I^0_\beta, \tilde{B} \rangle \rightarrow \langle I^0_\beta, B \rangle$ be the uncollapse of $H$.

(1) $\tilde{\pi}$ is an extension of $\pi$

Let $\tilde{\rho} = \text{sup}(\pi[\tilde{\rho}])$ and $\tilde{A} = A \cap J^X_{\tilde{\rho}}$. Then $\pi : \langle J^X_{\tilde{\rho}}, X \mid \tilde{\rho}, \tilde{A} \rangle \rightarrow \langle J^X_{\tilde{\rho}}, X \mid \tilde{\rho}, \tilde{A} \rangle$
\( \hat{\rho}, \hat{A} \) is \( \Sigma_0 \)-elementary, and by lemma 18, it is even \( \Sigma_1 \)-elementary. We have \( \operatorname{rng}(\pi) = H \cap J^X_\beta \). Obviously \( \operatorname{rng}(\pi) \subseteq H \cap J^X_\beta \). So let \( \rho \in H \cap J^X_\beta \). Then there is an \( i \in \omega \) and an \( x \in \operatorname{rng}(\pi) \) such that \( y \) is the unique \( y \in J^X_\beta \) that satisfies \( \langle I^0_\beta, B \rangle \models \varphi_i(y, x, p) \). So by definition of \( A, y \) is the unique \( y \in J^X_\beta \) such that \( A(i, (y, x)) \). But \( x \in \operatorname{rng}(\pi) \) and \( \pi : (J^X_\beta, X \upharpoonright \hat{\rho}, \hat{A}) \rightarrow (J^X_\beta, X \upharpoonright \hat{\rho}, \hat{A}) \) is \( \Sigma_\lambda \)-elementary. Therefore \( y \in \operatorname{rng}(\pi) \). So we have proved that \( H \) is an \( \epsilon \)-end-extension of \( \operatorname{rng}(\pi) \). Since \( \pi \) is the collapse of \( \operatorname{rng}(\pi) \) and \( \tilde{\pi} \) the collapse of \( H \), we obtain \( \pi \subseteq \tilde{\pi} \).

(2) \( \tilde{\pi} : \langle I^0_\beta, B \rangle \rightarrow \langle I^0_\beta, B \rangle \) is \( \Sigma_{m+1} \)-elementary

We must prove \( H \prec_{m+1} \langle I^0_\beta, B \rangle \). If \( m = 0 \), this is clear. So let \( m > 0 \) and let \( y \in \Sigma_{m+1} \)-definable in \( \langle I^0_\beta, B \rangle \) with parameters from \( \operatorname{rng}(\pi) \cup \{ p \} \). Then we have to show \( y \in H \). Let \( \varphi \) be a \( \Sigma_{m+1} \) formula and \( x_i \in \operatorname{rng}(\pi) \) such that \( y \) is uniquely determined by \( \langle I^0_\beta, B \rangle \models \varphi(y, x_i, p) \). Let \( h((i, x)) \approx h((i, x, p)) \). Then \( h[I^X_\beta] = J^X_\beta \) by the definition of \( p \). So there is a \( z \in J^X_\beta \) such that \( y = \hat{h}(z) \). If such a \( z \) lies in \( J^X_\beta \cap H \), then also \( y \in H \), since \( z, p \in H \prec \rho \langle I^0_\beta, B \rangle \). Let \( D = \text{dom}(h) \cap J^X_\beta \). Then it suffices to show

\[
(\ast) \quad (\exists z_0 \in D)(\forall z_1 \in D) \ldots \langle I^0_\beta, B \rangle \models \psi(h(z_1), h(z_1), z_1, x_i, p)
\]

for some \( z \in H \cap J^X_\beta \) where \( \psi \) is \( \Sigma_1 \) for even \( m \) and \( \Pi_1 \) for odd \( m \) such that \( \psi(y, x_i, p) \Leftrightarrow \langle I^0_\beta, B \rangle \models (\exists z_0)(\forall z_1) \ldots \psi(z_1, y, x_i, p) \). First, let \( m > 0 \) be even. Since \( A \) is the standard code, there is an \( i_0 \in \omega \) such that \( z \in D \Leftrightarrow A(i_0, x) \) holds for all \( z \in J^X_\beta \) and a \( j_0 \in \omega \) such that, for all \( z_1, z \in D, \langle I^0_\beta, B \rangle \models \psi(h(z_1), h(z), z_1, x_i, p) \) iff \( A(j_0, (z_1, z, x_i)) \). Thus (\( \ast \)) is, for \( z \in J^X_\beta \), equivalent with an obvious \( \Sigma_m \) formula. If \( m \) is odd, then write in (\( \ast \)) \( \ldots \langle I^0_\beta, B \rangle \models \neg(\psi(\ldots)) \). Then \( \neg(\psi) \in \Sigma_1 \) and we can proceed as above. Eventually \( \pi : \langle I^0_\beta, \tilde{A} \rangle \rightarrow \langle I^0_\beta, A \rangle \) is \( \Sigma_\lambda \)-elementary by the hypothesis and \( \pi \subseteq \tilde{\pi} \) by (1) – i.e. \( H \cap J^X_\beta \prec \rho \langle I^0_\beta, A \rangle \). Since there is a \( z \in J^X_\beta \) which satisfies (\( \ast \)) and \( x_i, p \in H \cap J^X_\beta \), there exists such a \( z \in H \cap J^X_\beta \).

Let \( H \prec I^0_\beta, A \rangle \). Then \( \pi \) has a \( \Sigma_1 \)-elementary extension \( \tilde{\pi} : \langle I^0_\beta, \tilde{B} \rangle \rightarrow \langle I^0_\beta, B \rangle \). So \( H \models \langle I^0_\beta, \tilde{A} \rangle \) for some \( \tilde{\rho} \) and \( \tilde{A} \). i.e. \( \text{Cond}_A(I^0_\beta) \).

(3) \( \tilde{A} = \{ (i, x) \mid i \in \omega \text{ and } x \in J^X_\beta \text{ and } \langle I^0_\beta, B \rangle \models \varphi_i(x, \tilde{x}^{-1}(p)) \} \)

Since \( \pi : \langle I^0_\beta, \tilde{A} \rangle \rightarrow \langle I^0_\beta, A \rangle \) is \( \Sigma_\lambda \)-elementary, \( \tilde{A}(i, x) \Leftrightarrow A(i, \pi(x)) \) for \( x \in J^X_\beta \).

Since \( A \) is the standard code of \( \langle I^0_\beta, B \rangle \), \( A(i, \pi(x)) \Leftrightarrow \langle I^0_\beta, B \rangle \models \varphi_i(x, \pi^{-1}(p)) \).

Finally, \( \langle I^0_\beta, B \rangle \models \varphi_i(x, p) \Leftrightarrow \langle I^0_\beta, B \rangle \models \varphi_i(x, \tilde{x}^{-1}(p)) \), because \( \pi : \langle I^0_\beta, B \rangle \rightarrow \langle I^0_\beta, B \rangle \) is \( \Sigma_1 \)-elementary.

(4) \( \tilde{\rho} \) is the projection of \( \langle I^0_\beta, \tilde{B} \rangle \)

By the definition of \( H, J^X_\beta = h_{\beta, B}[\omega \times (J^X_\beta \times \{ \tilde{x}^{-1}(p) \})] \). So \( f(i, x) \approx h_{\beta, B}(i, (x, \tilde{x}^{-1}(p))) \) is a \( \Sigma_1 \)-definable function such that \( f[J^X_\beta] = J^X_\beta \).

It remains to prove that \( \langle I^0_\beta, C \rangle \) is rudimentary closed for all \( C \in \Sigma_1((I^0_\beta, B) \cap \Psi(J^0_\beta)) \).

By the definition of \( H \), there exists an \( i \in \omega \) and \( y \in J^X_\beta \) such that \( x \in C \Leftrightarrow \langle I^0_\beta, B \rangle \models \varphi_i(x, y, \tilde{x}^{-1}(p)) \) for all \( x \in J^X_\beta \). Thus, by (3), \( x \in C \Leftrightarrow A(i, (x, y)) \). For \( u \in J^X_\beta \), let \( v = \{(i, (x, y)) \mid x \in u \} \). Then \( v \in J^X_\beta \) and \( A \cup v \in J^X_\beta \), because \( \langle I^0_\beta, A \rangle \) is rudimentary closed by the hypothesis. But
Since there were an this property. Suppose that \( C \subseteq \langle x, p \rangle \). Let \( \sigma \) be the standard parameter of \( n \). Call By lemma 17, \( \beta > \omega \). For \( n \) iterates this process. From lemma 19, we get by induction:

(1) There is a unique \( \bar{\beta} \geq \bar{\rho} \) such that \( \bar{\rho} \) is the n-th projectum and \( \bar{A} \) is the n-th code of \( \bar{\beta} \).

For \( k \leq n \) let

\[ x \in C \cap u \ \text{holds iff} \ (i, \langle x, y \rangle) \in \bar{A} \cap v. \] Finally, \( J^X_\rho \) is rudimentary closed and therefore \( C \cap u \subseteq J^X_\rho \).

(5) \( \bar{\pi}^{-1}(p) \) is the standard parameter of \( \langle I^0_{\beta}, \bar{B} \rangle \).

By the definition of \( H \), \( J^X_\rho = h_{\beta, \bar{B}}[\omega \times \{ J^X_\rho \times \{ \bar{\pi}^{-1}(p) \} \}] \) and, by (4), \( \bar{\rho} \) is the projectum of \( \langle I^0_{\beta}, \bar{B} \rangle \). So we just have to prove that \( \bar{\pi}^{-1}(p) \) is the least with this property. Suppose that \( \bar{p}' < \bar{\pi}^{-1}(p) \) had this property as well. Then there were an \( i \in \omega \) and an \( x \in J^X_\rho \) such that \( \bar{\pi}^{-1}(p) = h_{\beta, B}(i, \langle x, \bar{p}' \rangle) \). Since \( \bar{\pi} : \langle I^0_{\beta}, \bar{B} \rangle \to \langle I^0_{\rho}, \bar{B} \rangle \) is \( \Sigma_1 \)-elementary, we had \( p = h_{\beta, B}(i, \langle x, \bar{p}' \rangle) \) for \( p' = \pi(\bar{p}') < p \). And so also \( h_{\beta, B} \times \{ J^X_\rho \} = J^X_\rho \) that contradicts the definition of \( p \).

(6) Uniqueness

Assume \( \langle I^0_{\beta}, \bar{B}_0 \rangle \) and \( \langle I^0_{\beta}, \bar{B}_1 \rangle \) both have \( \bar{\rho} \) as projectum and \( \bar{A} \) as standard code. Let \( \bar{a}_i \) be the standard parameter of \( \langle I^0_{\beta}, \bar{B}_i \rangle \). Then, for all \( j \in \omega \) and \( x \in J^X_\rho \), \( \langle I^0_{\beta}, \bar{B}_0 \rangle \models \varphi_j(x, \bar{p}_0) \) iff \( \bar{A}(j, x) \models \varphi_j(x, \bar{p}_1) \). So \( \sigma(h_{\beta, 0}(j, \langle x, \bar{p}_0 \rangle)) \models h_{\beta, 1}(j, \langle x, \bar{p}_1 \rangle) \) defines an isomorphism \( \sigma : \langle I^0_{\beta}, \bar{B}_0 \rangle \cong \langle I^0_{\beta}, \bar{B}_1 \rangle \), because, for both \( i, h_{\beta, B} \times \{ J^X_\rho \} \) holds. But since both structures are transitive, \( \sigma \) must be the identity. Finally, let \( \bar{\pi}_0 : \langle I^0_{\beta}, \bar{B}_0 \rangle \to \langle I^0_{\beta}, \bar{B} \rangle \) and \( \bar{\pi}_1 : \langle I^0_{\beta}, \bar{B}_1 \rangle \to \langle I^0_{\beta}, \bar{B} \rangle \) be \( \Sigma_1 \)-elementary extensions of \( \pi \). Let \( \bar{p} \) be the standard parameter of \( \langle I^0_{\beta}, \bar{B} \rangle \). Then, for every \( y \in J^X_\rho \), there is an \( x \in J^X_\rho \) and a \( j \in \omega \) such that \( y = h_{\beta, B}(j, \langle x, \bar{p} \rangle) \). So \( \bar{\pi}_0(y) = h_{\beta, B}(j, \pi(x), \pi(p)) = \bar{\pi}_1(y) \). Thus \( \bar{\pi}_0 = \bar{\pi}_1 \). \( \square \)

To code the \( \Sigma_n \) information of \( I_\beta \) where \( \beta \in S^X \) in a structure \( \langle I^0_{\beta}, A \rangle \), one iterates this process.

For \( n \geq 0, \beta \in S^X \), let

\[
\begin{align*}
\rho^0 &= \beta, \ p^0 = \emptyset, \ A^0 = X_\beta \\
\rho^{n+1} &= \text{the projectum of} \ (I^0_{\rho^n}, A^n) \\
p^{n+1} &= \text{the standard parameter of} \ (I^0_{\rho^n}, A^n) \\
A^{n+1} &= \text{the standard code of} \ (I^0_{\rho^n}, A^n).
\end{align*}
\]

Call

\( \rho^n \) the n-th projectum of \( \beta \),

\( p^n \) the n-th (standard) parameter of \( \beta \),

\( A^n \) the n-th (standard) code of \( \beta \).

By lemma 17, \( A^n \subseteq J^X_\rho \) is \( \Sigma_n \)-definable over \( I_\beta \) and, for all \( m \geq 1 \),

\[ \Sigma_{n+m}(I_\beta) \cap \mathcal{P}(J^X_\rho) = \Sigma_m((I^0_{\rho^n}, A^n)). \]

From lemma 19, we get by induction:

For \( \beta > \omega, n \geq 1, m \geq 0 \), let \( \rho^n \) be the n-th projectum and \( A^n \) be the n-th code of \( \beta \). Let \( \langle I^0_{\rho}, \bar{A} \rangle \) be a rudimentary closed structure and \( \pi : \langle I^0_{\rho}, \bar{A} \rangle \to \langle I^0_{\rho^n}, A^n \rangle \) be \( \Sigma_m \)-elementary. Then:

(1) There is a unique \( \bar{\beta} \geq \bar{\rho} \) such that \( \bar{\rho} \) is the n-th projectum and \( \bar{A} \) is the n-th code of \( \bar{\beta} \).

For \( k \leq n \) let
\(p^k\) be the \(k\)-th projection of \(\beta\),
\(p^k\) the \(k\)-th parameter of \(\beta\),
\(A^k\) the \(k\)-th code of \(\beta\)
and
\(\bar{p}^k\) the \(k\)-th projection of \(\bar{\beta}\),
\(\bar{p}^k\) the \(k\)-th parameter of \(\bar{\beta}\),
\(\bar{A}^k\) the \(k\)-th code of \(\bar{\beta}\).

(2) There exists a unique extension \(\bar{\pi}\) of \(\pi\) such that, for all \(0 \leq k \leq n\),
\[\bar{\pi} \upharpoonright \bar{J}_p^X : \langle J_p^n \rangle \cup A^k \rightarrow \langle J_p^n \rangle \cup A^k\]
is \(\Sigma_{m+n-k}\)-elementary and this is \(\Sigma_{m+n-k}\)-elementary.

\[\bar{\pi}(\bar{p}^k) = p^k.\]

**Lemma 20**

Let \(\omega < \beta \in S^X\). Then all projecta of \(\beta\) exist.

**Proof** by induction on \(n\). That \(\rho^0\) exists is clear. So suppose that the first projecta \(\rho^0, \ldots, \rho^{n-1}, \rho := \rho^n\), the parameters \(\rho^0, \ldots, \rho^n\) and the codes \(A^0, \ldots, A^{n-1}, A := A^n\) of \(\beta\) exist. Let \(\gamma \in \text{Lim}\) be minimal such that there is some over \(\langle I_\rho^0, A \rangle\)
\(\Sigma_1\)-definable function \(f\) such that \(f[J_\gamma^X] = J_\rho^X\). Let \(C \in \Sigma_1(\langle I_\rho^0, A \rangle) \cup \mathfrak{H}(J_\gamma^X)\).
We have to prove that \(\langle I_\rho^0, C \rangle\) is rudimentary closed. If \(\gamma = \omega\), then \(J_\gamma^X = H_\omega\),
and this is obvious. If \(\gamma > \omega\), then \(\gamma \in \text{Lim}\) by the definition of \(\gamma\). Then it
suffices to show \(C \cap J_\delta^X \subseteq J_\gamma^X\) for \(\delta \in \text{Lim} \cap \gamma\). Let \(B := C \cap J_\delta^X\) be definable
over \(\langle I_\rho^0, A \rangle\) with parameter \(q\). Since obviously \(\gamma \leq \rho\), \(C \cap J_\delta^X\) is \(\Sigma_n\)-definable
over \(I_\beta\) with parameters \(p^0, \ldots, p^n\) by lemma 17. So let \(\psi\) be a \(\Sigma_n\) formula
such that \(x \in C \iff I_\beta \models \psi(x, p^0, \ldots, p^n, q)\). Let
\[
\begin{align*}
H_{n+1} &:= h_{p^n.\rho^n.\Sigma^n}[\omega \times (J_\rho^X \times \{q\})] \\
H_n &:= h_{p^{n-1}.\rho^{n-1}[\omega \times (H_{n+1} \times \{p^n\})]} \\
H_{n-1} &:= h_{p^{n-2}.\rho^{n-2}[\omega \times (H_{n-1} \times \{p^{n-1}\})]} \\
\end{align*}
\]
etc.

Since \(L[X]\) has condensation, there is an \(I_\mu\) such that \(H_1 \cong I_\mu\). Let \(\pi\) be the uncollapse of \(H_1\). Then \(\pi\) is the extension of the collapse of \(H_{n+1}\) defined in
the proof of lemma 19. Therefore it is \(\Sigma_{n+1}\)-elementary. Since \(B \subseteq J_\delta^X\) and
\(\pi \upharpoonright J_\lambda^X = \text{id} \upharpoonright J_\lambda^X\), we get \(x \in B \iff I_\mu \models \psi(x, \pi^{-1}(p^1), \ldots, \pi^{-1}(p^n), \pi^{-1}(q))\).
So \(B\) is indeed already \(\Sigma_n\)-definable over \(I_\mu\). Thus \(B \subseteq J_\rho^X\) by lemma 8. But now we are done because \(\mu < \rho\). For, if
\[
\begin{align*}
h_{n+1}((i, x)) &= h_{p^n.\rho^n.\Sigma^n}(i, (x, p^n)) \\
h_n((i, x)) &= h_{p^{n-1}.\rho^{n-1}[i, (x, p^{n-1})]} \\
\end{align*}
\]
etc.

then the function \(h = h_1 \circ \ldots \circ h_{n+1}\) is \(\Sigma_{n+1}\)-definable over \(I_\beta\). Thus the function
\(h = \pi[h \cap (H_1 \times H_1)]\) is \(\Sigma_{n+1}\)-definable over \(I_\mu\) and \(h[J_\gamma^X] = J_\lambda^X\).
So \(h \cap (J_\lambda^X)^2\) is \(\Sigma_1\)-definable over \(\langle I_\rho^0, A \rangle\) by lemma 17 and lemma 19. And by the definition of \(\gamma\),
there is an \(\omega\)-definable function \(f\) such that \(f[J_\gamma^X] = J_\rho^X\). So
if we had \(\mu \geq \rho\), then \(f \circ h\) was an over \(\langle I_\rho^0, A \rangle\) \(\Sigma_1\)-definable function such that
\(f \circ h)[J_\lambda^X] = J_\rho^X\). That contradicts the minimality of \(\gamma\). \(\square\)

Let \(\omega < \nu \in S^X\), \(p^n\) the \(n\)-th projection of \(\nu\), \(p^n\) the \(n\)-th parameter and \(A^n\)
the \(n\)-th Code. Let
\[
\begin{align*}
h_{n+1}((i, x)) &= h_{p^n.\rho^n.\Sigma^n}(i, x) \\
h_n((i, x)) &= h_{p^{n-1}.\rho^{n-1}[i, (x, p^n)]} \\
\end{align*}
\]
Let \( \gamma \in \text{Lim} \) such that there is a over \( I_\beta \Sigma_n \)-definable function \( f \) such that \( f[J^n X] = J^n X \), \( n \geq 1 \).

(1) By the definition of the \( n \)-th projectum, there is an over \( \{I^n_\rho, \rho^n_A \} \Sigma_1 \)-definable \( f^n \) such that \( f^n[J^n X] = J^n X \), an over \( \{I^n_\rho, \rho^n_A \} \Sigma_1 \)-definable \( f^n \) such that \( f^n[J^n X] = J^n X \), etc.

(2) By the definition of the \( n \)-th projectum, \( \{I^n_\rho, \rho^n_A \} \Sigma_1 \)-definable function \( f \) such that \( f[J^n X] = J^n X \), then \( g := f \cap (J^n X ρ^n A) \) would be an over \( \{I^n_\rho, \rho^n_A \} \Sigma_1 \)-definable function such that \( g[J^n X] = J^n X \), but this is impossible.

Proof:

(1) By the definition of the \( n \)-th projectum, \( \{I^n_\rho, \rho^n_A \} \Sigma_1 \)-definable function \( f \) such that \( f[J^n X] = J^n X \). For, suppose there is no such \( \rho \) such that such an \( f \), \( f[J^n X] = J^n X \), exists. Then the proof of lemma 16 provides a contradiction. So if there were such a \( \gamma < \rho^n \) such that \( f[J^n X] = J^n X \), then \( g := f \cap (J^n X ρ^n A) \) would be an over \( \{I^n_\rho, \rho^n_A \} \Sigma_1 \)-definable function such that \( g[J^n X] = J^n X \). But this is impossible.

(2) By the definition of the \( n \)-th projectum, \( \{I^n_\rho, \rho^n_A \} \Sigma_1 \)-definable function \( f \) such that \( f[J^n X] = J^n X \), then \( g := f \cap (J^n X ρ^n A) \) would be an over \( \{I^n_\rho, \rho^n_A \} \Sigma_1 \)-definable function such that \( g[J^n X] = J^n X \), but this is impossible.

Assume \( \gamma \) were a larger ordinal in \( \text{Lim} \) having this property. Let \( f \) be, by (1), an over \( I_\beta \Sigma_n \)-definable function such that \( f[J^n X] = J^n X \). Set \( C = \{u \in J^n X \mid u \notin f(\{f\})\} \). Then \( C \) is \( \Sigma_n \)-definable over \( I_\beta \) and \( C \subseteq J^n X \). So \( J^n X, C \) was rudimentary closed. And therefore \( C = C \cap J^n X \subseteq J^n X \) and \( C = f(u) \) for some \( u \in J^n X \). But this implies the contradiction that \( u \in f(u) \Leftrightarrow u \in C \Leftrightarrow u \notin f(u) \).

(3) Let \( \rho := \rho^n \) and \( f \) by (1) an over \( I_\beta \Sigma_n \)-definable function such that \( f[J^n X] = J^n X \). Let \( j \) be an over \( I^n_\rho \Sigma_1 \)-definable function from \( \rho \) onto \( J^n X \). Let \( C = \{\nu \in \rho \mid \nu \notin f \circ j(\nu)\} \). Then \( C \) is an over \( I_\beta \Sigma_n \)-definable subset of \( \rho \). If \( C \in J^n X \), then there would be a \( \nu \in \rho \) such that \( C = f \circ j(\nu) \), and we had the contradiction \( \nu \in C \Leftrightarrow \nu \notin f \circ j(\nu) \Leftrightarrow \nu \notin C \). Thus \( \langle \rangle \cap \Sigma_n[I_\beta] \notin J^n X \). But if \( \gamma \in \text{Lim} \cap \rho \) and \( D \in \mathfrak{P}(\gamma) \cap \Sigma_n(I_\beta) \), then \( D = D \cap J^n X \subseteq J^n X \). So \( \mathfrak{P}(\gamma) \cap \Sigma_n(I_\beta) \subseteq J^n X \).
3 Morasses

Let \( \omega_1 \leq \beta, S = \text{Lim} \cap \omega_{1+\beta} \) and \( \kappa := \omega_{1+\beta} \).

We write \( \text{Card} \) for the class of cardinals and \( \text{RCard} \) for the class of regular cardinals.

Let \( \prec \) be a binary relation on \( S \) such that:

(a) If \( \nu < \tau \), then \( \nu < \tau \).

- For all \( \nu \in S - \text{RCard} \), \( \{ \tau \mid \nu \prec \tau \} \) is closed.

- For \( \nu \in S - \text{RCard} \), there is a largest \( \mu \) such that \( \nu \leq \mu \).

Let \( \mu_\nu \) be this largest \( \mu \) with \( \nu \leq \mu \).

Let

\[
\nu \subseteq \tau \iff \nu \in \text{Lim}(\{\delta \mid \delta \prec \tau\}) \cup \{\delta \mid \delta \leq \tau\}.
\]

Hence, if \( \nu \notin \text{RCard} \) is a successor in \( \subseteq \), then \( \mu_\nu \) is the largest \( \mu \) such that \( \nu \subseteq \mu \).

To see this, let \( \mu^*_\nu \) be the largest \( \mu \) such that \( \nu \subseteq \mu \). It is clear that \( \mu_\nu \leq \mu^*_\nu \), since \( \nu \leq \mu \) implies \( \nu \subseteq \mu \). So assume that \( \mu_\nu < \mu^*_\nu \). Then \( \nu \not\in \mu^*_\nu \) by the definition of \( \mu_\nu \).

Hence \( \nu \in \text{Lim}(\{\delta \mid \delta < \mu^*_\nu\}) \) and \( \nu \in \text{Lim}(\{\delta \mid \delta \subseteq \mu^*_\nu\}) \).

Therefore, \( \nu \in \text{Lim}(\subseteq) \) since \( \subseteq \) is a tree. That contradicts our assumption that \( \nu \) is a successor in \( \subseteq \).

For \( \alpha \in S \), let \( |\alpha| \) be the rank of \( \alpha \) in this tree. Let

\[
S^+ := \{ \nu \in S \mid \nu \text{ is a successor in } \subseteq \}
\]

\[
S^0 := \{ \alpha \in S \mid |\alpha| = 0 \}
\]

\[
\hat{S}^+ := \{ \mu_\tau \mid \tau \in S^+ - \text{RCard} \}
\]

\[
\hat{S} := \{ \mu_\tau \mid \tau \in S - \text{RCard} \}.
\]

Let \( S_\alpha := \{ \nu \in S \mid \nu \text{ is a direct successor of } \alpha \text{ in } \subseteq \}. \) For \( \nu \in S^+ \), let \( \alpha_\nu \) be the direct predecessor of \( \nu \) in \( \subseteq \). For \( \nu \in S^0 \), let \( \alpha_\nu := 0 \). For \( \nu \notin S^+ \cup S^0 \), let \( \alpha_\nu := \nu \).

(c) For \( \nu, \tau \in (S^+ \cup S^0) - \text{RCard} \) such that \( \alpha_\nu = \alpha_\tau \), suppose:

\[
\nu < \tau \implies \mu_\nu < \tau.
\]

For all \( \alpha \in S \), suppose:

(d) \( S_\alpha \) is closed

(e) \( \text{card}(S_\alpha) \leq \alpha^+ \)

\( \text{card}(S_\alpha) \leq \text{card}(\alpha) \) if \( \text{card}(\alpha) < \alpha \)

(f) \( \omega_1 = \max(S^0) = \text{sup}(S^0 \cap \omega_1) \)

\( \omega_{1+i+1} = \max(S_{\omega_{1+i}}) = \text{sup}(S_{\omega_{1+i}} \cap \omega_{1+i+1}) \text{ for all } i < \beta \).

Let \( D = \{ D_\nu \mid \nu \in \hat{S} \} \) be a sequence such that \( D_\nu \subseteq J^D_\nu \). To simplify matters, my definition of \( J^D_\nu \) is such that \( J^D_\nu \cap \text{On} = \nu \) (see section 3 or [SchZe]).

Let an \( (S, \prec, D) \)-maplet \( f \) be a triple \( (\varnothing, |f|, \nu) \) such that \( \varnothing, \nu \in S - \text{RCard} \) and \( |f| : J^D_\varnothing \to J^D_{\mu_\varnothing} \).

Let \( f = (\varnothing, |f|, \nu) \) be an \( (S, \prec, D) \)-maplet. Then we define \( d(f) \) and \( r(f) \) by \( d(f) = \nu \) and \( r(f) = \nu \). Set \( f(x) := |f|(x) \) for \( x \in J^D_{\mu_\varnothing} \) and \( f(\mu_\varnothing) := \mu_\varnothing \).
But $\text{dom}(f)$, $\text{rng}(f)$, $f \upharpoonright X$, etc. keep their usual set-theoretical meaning, i.e. $\text{dom}(f) = \text{dom}([f])$, $\text{rng}(f) = \text{rng}([f])$, $f \upharpoonright X = [f] \upharpoonright X$, etc.

For $\bar{x} \subseteq \mu_\nu$, let $f^{(\tau)} = (\bar{x}, [f] \upharpoonright J_{\mu_\nu}^f, \tau)$ where $\tau = f(\bar{x})$. Of course, $f^{(\tau)}$ needs not to be a maplet. The same is true for the following definitions. Let $f^{-1} = (\nu, [f]^{-1}, \bar{\nu})$. For $g = (\nu, [g], \nu')$ and $f = (\bar{\nu}, [f], \nu)$, let $g \circ f = (\bar{\nu}, [g] \circ [f], \nu')$. If $g = (\nu', [g], \nu)$ and $f = (\bar{\nu}, [f], \nu)$ such that $\text{rng}(f) \subseteq \text{rng}(g)$, then set $g^{-1}f = (\bar{\nu}, [g]^{-1} \upharpoonright [f], \nu')$.

Finally set $id_\nu = (\nu, id \upharpoonright J_{\mu_\nu}^f, \nu)$.

Let $\mathfrak{F}$ be a set of $(S, \subset, D)$-maplets $f = (\bar{\nu}, [f], \nu)$ such that the following holds:

1. $f(\bar{\nu}) = \nu$, $f(\alpha_\nu) = \alpha_\nu$ and $[f]$ is order-preserving.
2. For $f \neq id_\nu$, there is some $\beta \subseteq \alpha_\nu$ such that $f \upharpoonright \beta = id \upharpoonright \beta$ and $f(\beta) > \beta$.
3. If $\bar{\tau} \in \nu_+ \nu$ and $\bar{\nu} \subseteq \bar{\tau} \subseteq \nu_\mu$, then $f^{(\tau)} \in \mathfrak{F}$.
4. If $f, \bar{g} \in \mathfrak{F}$ and $d(g) = r(f)$, then $g \circ f \in \mathfrak{F}$.

We write $f : \nu \Rightarrow \nu$ if $f = (\bar{\nu}, [f], \nu) \in \mathfrak{F}$. If $f \in \mathfrak{F}$ and $r(f) = \nu$, then we write $f \Rightarrow \nu$. The uniquely determined $\beta$ in (1) shall be denoted by $\beta(f)$.

Say $f \in \mathfrak{F}$ is minimal for a property $P(f)$ if $P(f)$ holds and $P(g)$ implies $g^{-1}f \in \mathfrak{F}$.

Let $f_{(u, x, \nu)} = \text{the unique minimal } f \in \mathfrak{F} \text{ for } f \Rightarrow \nu \text{ and } u \cup \{x\} \subseteq \text{rng}(f)$, if such an $f$ exists. The axioms of the morass will guarantee that $f_{(u, x, \nu)}$ always exists if $\nu \in S - \text{RCard}^{L_{\omega}.[D]}$. Therefore, we will always assume and explicitly mention that $\nu \in S - \text{RCard}^{L_{\omega}.[D]}$ when $f_{(u, x, \nu)}$ is mentioned.

Say $\nu \in S - \text{RCard}^{L_{\omega}.[D]}$ is independent if $d(f_{(\beta_0, 0, \nu)}) < \alpha_\nu$ holds for all $\beta < \alpha_\nu$.

For $\tau \subseteq \nu \in S - \text{RCard}^{L_{\omega}.[D]}$, say $\nu$ is $\xi$-dependent on $\tau$ if $f_{(\alpha_\tau, \xi, \nu)} = id_\nu$.

For $f \in \mathfrak{F}$, let $\lambda(f) := \sup\{f[d(f)]\}$.

For $\nu \in S - \text{RCard}^{L_{\omega}.[D]}$ let

$$C_{\nu} = \{\lambda(f) < \nu \mid f \Rightarrow \nu\}$$

$$\Lambda(x, \nu) = \{\lambda(f_{(x, \nu, \nu)} < \nu \mid \beta < \nu\}.$$ 

It will be shown that $C_{\nu}$ and $\Lambda(x, \nu)$ are closed in $\nu$.

Recursively define a function $q_\nu : k_\nu + 1 \rightarrow \text{On}$, where $k_\nu \in \omega$:

$q_\nu(0) = 0$

$q_\nu(k + 1) = \max(\Lambda(q_\nu \upharpoonright (k + 1), \nu))$

if $\max(\Lambda(q_\nu \upharpoonright (k + 1), \nu))$ exists. The axioms will guarantee that this recursion breaks off (see lemma 4 below), i.e. there is some $k_\nu$ such that either

$$\Lambda(q_\nu \upharpoonright (k_\nu + 1), \nu) = \emptyset$$

or

$$\Lambda(q_\nu \upharpoonright (k_\nu + 1), \nu) \text{ is unbounded in } \nu.$$ 

Define by recursion on $1 \leq n \in \omega$, simultaneously for all $\nu \in S - \text{RCard}^{L_{\omega}.[D]}$, $\beta \in \nu$ and $x \in J_{\mu_\nu}$ the following notions:
For $i$ (CP1 – first continuity principle)

$\tau(n, \nu) = \text{the least } \tau \in S^0 \cup S^+ \cup \hat{S} \text{ such that for some } x \in J^D_{\mu_\nu}$

$f_{(\alpha, x, \nu)}^n = id_\nu$

$x(n, \nu) = \text{the least } x \in J^D_{\mu_\nu} \text{ such that } f_{(\alpha, x, \nu)}^n = id_\nu$

$K^\nu_n = \{d(f_{(\beta, x, \nu)}^n) < \alpha_{\tau(n, \nu)} | \beta < \nu\}$

$f \Rightarrow \nu \text{ if } f \Rightarrow \nu \text{ and for all } 1 \leq m < n$

$rng(f) \cap J^D_{\alpha_{\tau(m, \nu)} \bar{\mathcal{A}}} \prec_1 (J^D_{\alpha_{\tau(m, \nu)}, D \upharpoonright \alpha_{\tau(m, \nu)}}, K^m_{\nu})$

$x(m, \nu) \in rng(f)$

$f_{(a, \nu)}^n = \text{the minimal } f \Rightarrow \nu \text{ such that } u \subseteq rng(f)$

$f_{(\beta, x, \nu)}^n = f_{(\beta \cup (x)), \nu}$

$f : \bar{\nu} \Rightarrow \nu \Leftrightarrow f \Rightarrow \nu \text{ and } f : \bar{\nu} \Rightarrow \nu.$

Here definitions are to be understood in Kleene’s sense, i.e., that the left side is defined iff the right side is defined and in that case, both are equal.

Let

$n_\nu = \text{the least } n \text{ such that } f_{(\gamma, x, \mu_\nu)}(\gamma, x, \mu_\nu) \text{ is confinal in } \nu \text{ for some } x \in J^D_{\mu_\nu}$

$x_\nu = \text{the least } x \text{ such that } f_{(\alpha, x, \mu_\nu)} = id_{\mu_\nu}$

Let

$\alpha^*_\nu = \alpha_\nu \text{ if } \nu \in S^+$

$\alpha^*_\nu = \sup\{\alpha < \nu | \beta(f_{(\alpha, x, \mu_\nu)}^n) = \alpha\} \text{ if } \nu \notin S^+.$

Let $P_\nu := \{x \mid \nu \subseteq \tau \subseteq \mu_\nu, \tau \in S^+ \} \cup \{x_\nu\}$.

We say that $\mathcal{M} = (S, <, \bar{\mathcal{A}}, D)$ is an $(\omega_1, \beta)$-morass if the following axioms hold:

(MP – minimum principle)
If $\nu \in S - RCard_{\mu_\nu}[D]$ and $x \in J^D_{\mu_\nu}$, then $f_{(0, x, \nu)}$ exists.

(LP1 – first logical preservation axiom)
If $f : \bar{\nu} \Rightarrow \nu$, then $|f| : (J^D_{\mu_\nu}, D \upharpoonright \mu_\nu) \rightarrow (J^D_{\mu_\nu}, D \upharpoonright \mu_\nu)$ is $\Sigma_1$-elementary.

(LP2 – second logical preservation axiom)
Let $f : \bar{\nu} \Rightarrow \nu$ and $f(\bar{x}) = x$. Then $f \upharpoonright J^D_{\nu} : (J^D_{\nu}, D \upharpoonright \nu, \Lambda(\bar{x}, \bar{\nu})) \rightarrow (J^D_{\nu}, D \upharpoonright \nu, \Lambda(x, \nu))$ is $\Sigma_0$-elementary.

(CP1 – first continuity principle)
For $i \leq j < \lambda$, let $f_i : \nu_i \Rightarrow \nu$ and $g_{ij} : \nu_i \Rightarrow \nu_j$ such that $g_{ij} = f_j^{-1}f_i$. Let $\langle g_i \mid i < \lambda \rangle$ be the transitive, direct limit of the directed system $\langle g_{ij} \mid i \leq j < \lambda \rangle$ and $hg_i = f_i$ for all $i < \lambda$. Then $g_i, h \in \bar{\mathcal{A}}$. 

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(CP2 – second continuity principle) Let \( f : \nu \mapsto \nu \) and \( \lambda = \sup(f[\nu]) \). If, for some \( \lambda, h : \langle J^D, D \rangle \to \langle J^D, D \upharpoonright \lambda \rangle \) is \( \Sigma_1 \)-elementary and \( \text{rng}(f \upharpoonright J^D) \subseteq \text{rng}(h) \), then there is some \( g : \lambda \mapsto \lambda \) such that \( g \upharpoonright J^D = h \).

(DF – definability axiom) If \( \nu \in S - \text{RCard}^{L_\omega[D]} \) is \( \eta \)-dependent on \( \tau \subseteq \nu, \tau \in S^+ \), \( f : \nu \mapsto \nu, f(\hat{\tau}) = \tau \) and \( \eta \in \text{rng}(f) \), then \( f(\hat{\tau}) : \hat{\tau} \mapsto \tau \).

(DF) is uniformly definable over \( S \).

(DP3 – third dependency axiom) For \( \nu \in S - \text{RCard}^{L_\omega[D]} \) and \( 1 \leq n \in \omega \), the following holds:

(a) If \( f^n_{(\alpha, \tau, \nu)} = \text{id}_\nu, \tau \in S^+ \cup S^0 \) and \( \tau \subseteq \nu \), then \( \mu_\nu = \mu_\tau \).

(b) If \( \beta < \alpha(\tau, \nu) \), then also \( d(f^n_{(\beta, x_{(\tau, \nu)}, \nu)}) < \alpha(\tau, \nu) \).

This finishes the definition of an \( (\omega_1, \beta) \)-morass.

A consequence of the axioms is \((\times)\) by \( [\text{Irr2}] \):

**Theorem**

\[
\{ \langle z, \tau, x, f_0(z, \tau)(x) \rangle \mid \tau < \nu, \mu_\tau = \nu, z \in J^D_{\mu_\nu}, x \in \text{dom}(f_0(z, \tau)) \} \\
\cup \{ \langle z, x, f_0(z, \nu)(x) \rangle \mid \mu_\nu = \nu, z \in J^D_{\mu_\nu}, x \in \text{dom}(f_0(z, \nu)) \} \\
\cup (\exists \nu^2)
\]

is for all \( \nu \in S \) uniformly definable over \( \langle J^D_{\nu}, D \upharpoonright \nu, D_{\nu} \rangle \).

A structure \( \mathfrak{M} = \langle S, \prec, \vec{F}, D \rangle \) is called an \( \omega_1 + \beta \)-standard morass if it satifies all axioms of an \( (\omega_1, \beta) \)-morass except (DF) which is replaced by:
If $\gamma$ is regulat in $\nu$ ⊈ $\nu$, then $\sigma_{(x, y)}(i) = \nu \in J^D_{\nu, D}$. If it is bounded, then $D_{\nu} = \{\nu \in J^D_{\nu, D} : \nu \in \text{dom}(\sigma_{(z, y)})\}$. Now, I am going to construct a $\kappa$-standard morass. Let $\beta(\nu)$ be the least $\beta$ such that $J^X_{\beta + 1} \models \nu$ singular.

Let $L^X_{\nu}[X]$ satisfy amenability, condensation and coherence such that $S^X = \{\beta(\nu) : \nu \text{ singular in } L^X_{\nu}[X]\}$ and $\text{Card}^L_{\nu}[X] = \text{Card}(\nu)$. Let

$$E = \text{Lim} - \text{RCard}^{L_{\nu}[X]}.$$ 

For $\nu \in E$, let $\beta(\nu) = \text{the least } \beta \text{ such that there is a cofinal } f : a \to \nu \in \text{Def}(I_{\beta}) \text{ and } a \subseteq \nu' < \nu$

$$\begin{align*}
n(\nu) &= \text{the least } n \geq 1 \text{ such that such an } f \text{ is } \Sigma_n \text{-definable over } I_{\beta(\nu)} \\
p(\nu) &= (n(\nu) - 1)\text{-th projectum of } I_{\beta(\nu)} \\
A_{\nu} &= (n(\nu) - 1)\text{-th standard code of } I_{\beta(\nu)} \\
\gamma(\nu) &= \text{the } n(\nu)\text{-th projectum of } I_{\beta(\nu)}.
\end{align*}$$

If $\nu \in S^+ - \text{Card}$, then the $n(\nu)\text{-th projectum } \gamma(\nu)$ of $\beta(\nu)$ is less or equal $\alpha_{\nu} := \text{the largest cardinal in } I_{\nu}$. Since $\alpha_{\nu}$ is the largest cardinal in $I_{\nu}$, there is, by definition of $\beta(\nu)$ and $n(\nu)$, some over $I_{\beta(\nu)} \Sigma_{n(\nu)}\text{-definable function } f \text{ such that } f(\alpha_{\nu}) \text{ is cofinal in } \nu$. But, since $\nu$ is regular in $\beta(\nu)$, $f$ cannot be an element of $J^X_{\beta(\nu)}$. So $\mathfrak{P}(\nu \times n) \cap \Sigma_{n(\nu)}(I_{\beta(\nu)}) \not\subseteq J^X_{\beta(\nu)}$. By lemma 14, also $\mathfrak{P}(\nu) \cap \Sigma_{n(\nu)}(I_{\beta(\nu)}) \not\subseteq J^X_{\beta(\nu)}$. Using lemma 21 (3), we get $\gamma < \nu$. I.e. there is an over $I_{\beta(\nu)} \Sigma_{n(\nu)}\text{-definable function } g \text{ such that } g(\nu) = J^X_{\beta(\nu)}$. On the other hand, there is, for every $\tau < \nu$ in $J^X_{\nu}$, a surjection from $\alpha_{\nu}$ onto $\tau$, because $\alpha_{\nu}$ is the largest cardinal in $I_{\nu}$. Let $f_{\tau}$ be the <-$\nu$-least such. Define $j_{1}(\sigma, \tau) = f_{f(\tau)}(\sigma)$ for $\sigma, \tau < \nu$. Then $j_{1}$ is $\Sigma_{n(\nu)}\text{-definable over } I_{\beta(\nu)}$ and $j_{1}[\alpha_{\nu} \times \alpha_{\nu}] = \nu$. By lemma 15, we obtain an
over \( I_{\beta(\nu)} \Sigma_{n(\nu)} \)-definable function \( j_2 \) from a subset of \( \alpha_\nu \) onto \( \nu \). Thus \( g \circ j_2 \) is an over \( I_{\beta(\nu)} \Sigma_{n(\nu)} \)-definable map such that \( g \circ j_2[\alpha_\nu] = J^X_{\beta(\nu)} \).

Moreover, \( \alpha_\nu < \nu \leq \rho(\nu) \): By definition of \( \rho(\nu) \), there is an over \( I_{\beta(\nu)} \Sigma_{n(\nu)-1} \)-definable function \( f \) such that \( f[\rho(\nu)] = \beta(\nu) \) if \( n(\nu) > 1 \). But \( \nu \) is \( \Sigma_{n(\nu)-1} \)-regular over \( I_{\beta(\nu)} \). Thus \( \nu \leq \rho(\nu) \). If \( n(\nu) = 1 \), then \( \rho(\nu) = \beta(\nu) \geq \nu \).

By the first inequality, there is a \( q \) such that every \( x \in J^X_{\rho(\nu)} \) is \( \Sigma_1 \)-definable in \( \langle I^0_{\rho(\nu)}, A_\nu \rangle \) with parameters from \( \alpha_\nu \cup \{ q \} \). Let \( p_\nu \) be the \( \langle \rho(\nu) \rangle \)-least such.

Obviously, \( p_\tau \leq p_\nu \) if \( \nu \subseteq \tau \subseteq \mu_\nu \).

Thus \( P_\nu := \{ p_\tau \mid \nu \subseteq \tau \subseteq \mu_\nu, \tau \in S^+ \} \) is finite.

Now, let \( \nu \in E - S^+ \). By definition of \( \beta(\nu) \), there exists no cofinal \( f : a \to \nu \) in \( J^X_\beta \) such that \( a \subseteq [\nu'] < \nu \). So \( \mathcal{P}(\nu \times \nu) \cap \Sigma_{n(\nu)}(I_{\beta(\nu)}) \subseteq J^X_{\beta(\nu)} \). Then, by lemma 14, \( \mathcal{P}(\nu) \cap \Sigma_{n(\nu)}(I_{\beta(\nu)}) \subseteq J^X_{\beta(\nu)} \). Hence, by lemma 21 (3),

\[
\gamma(\nu) \leq \nu.
\]

Assume \( \rho(\nu) < \nu \). Then there was an over \( I_{\beta(\nu)} \Sigma_{n(\nu)-1} \)-definable \( f \) such that \( f[\rho(\nu)] = \nu \). But this contradicts the definition of \( n(\nu) \). So

\[
\nu \leq \rho(\nu).
\]

Using lemma 21 (1), it follows from the first inequality that there is some over \( I_{\beta(\nu)} \Sigma_{n(\nu)} \)-definable function \( f \) such that \( f[J^X_\nu] = J^X_{\beta(\nu)} \). So there is a \( p \in J^X_{\rho(\nu)} \) such that every \( x \in J^X_{\rho(\nu)} \) is \( \Sigma_1 \)-definable in \( \langle I^0_{\rho(\nu)}, A_\nu \rangle \) with parameters from \( \nu \cup \{ p \} \). Let \( p_\nu \) be the least such.

Let

\[
\alpha^*_\nu = \sup \{ \alpha < \nu \mid h_{\rho(\nu), A_\nu}[\omega \times (J^X_\alpha \times \{ p_\nu \})] \cap \nu = \alpha \}.
\]

Then \( \alpha^*_\nu < \nu \) because, by definition of \( \beta(\nu) \), there exists a \( \nu' < \nu \) and a \( p \in J^X_{\rho(\nu)} \) such that \( h_{\rho(\nu), A_\nu}[\omega \times (J^X_\nu \times \{ p \})] \cap \nu \) is cofinal in \( \nu \). But \( p \) is in \( h_{\rho(\nu), A_\nu}[\omega \times (J^X_\nu \times \{ p_\nu \})] \). So there is an \( \alpha < \nu \) such that \( h_{\rho(\nu), A_\nu}[\omega \times (J^X_\alpha \times \{ p_\nu \})] \cap \nu \) is cofinal in \( \nu \). Thus \( \alpha^*_\nu < \alpha < \nu \).

If \( \nu \in S^+ \), then we set \( \alpha^*_{\nu} := \alpha_\nu \).

For \( \nu \in E \), let \( f : \nu \Rightarrow \nu \) iff, for some \( f^* \),

1. \( f = \langle \nu, f^* | J^{P_\nu}_\nu, \nu \rangle \),
2. \( f^* : I_{\mu_\nu} \to I_{\mu_\nu} \) is \( \Sigma_{n(\nu)} \)-elementary,
3. \( \alpha^*_\nu, p_\nu, \alpha^*_\nu, P_\nu \in \text{rng}(f^*) \),
4. \( \nu \in \text{rng}(f^*) \) if \( \nu < \mu_\nu \),
5. \( f(\bar{\nu}) = \nu \) and \( \bar{\nu} \in S^+ \iff \nu \in S^+ \).

By this, \( \mathfrak{F} \) is defined.

Set \( D = X \).

Let \( P_{\nu}^* \) be minimal such that \( h^{\rho(\nu)-1}_{\mu_\nu}(i, P_{\nu}^*) = P_\nu \) for an \( i \in \omega \).

Let \( \alpha_{\mu_\nu}^* \) be minimal such that \( h^{\rho(\nu)-1}_{\mu_\nu}(i, \alpha_{\mu_\nu}^*) = \alpha_{\mu_\nu}^* \) for some \( i \in \omega \).

Set

\[
\nu^* = \emptyset \text{ if } \nu = \rho(\nu)
\]
\[ \nu^* = \nu \text{ if } \nu < \rho(\nu). \]

For \( \tau \in On \), let \( S_\tau \) be defined as in lemma 10. For \( \tau \in On \), \( E_\tau \subseteq S_\tau \) and a \( \Sigma_0 \) formula \( \varphi \), let

\[ h^\varphi_{\tau,E}(x_1, \ldots, x_m) \]

the least \( x_0 \in S_\tau \) w.r.t. the canonical well-ordering such that

\[ \langle S_\tau, E_\tau \rangle \models \varphi(x_1) \]

if such an element exists,

and

\[ h^\varphi_{\tau,E}(x_1, \ldots, x_m) = \emptyset \text{ else.} \]

For \( \tau \in On \) such that \( \nu^*, \alpha^*_{\mu}, \omega^*_{\mu}, P^*_{\nu} \in S_\tau \), let \( H_\nu(\alpha, \tau) \) be the closure of

\[ S_\alpha \cup \{ \nu^*, \alpha^*_{\mu}, \omega^*_{\mu}, P^*_{\nu} \} \]

under all \( h^\varphi_{\tau,S_\tau \cap A_\nu \cap S_\tau} \). Then \( H_\nu(\alpha, \tau) \preceq \langle S_\tau, X \cap S_\tau, A_\nu \cap S_\tau, \{ \nu^*, \alpha^*_{\mu}, \omega^*_{\mu}, P^*_{\nu} \} \rangle \) by the definition of \( h^\varphi_{\tau,S_\tau \cap A_\nu \cap S_\tau} \). Let \( M_\nu(\alpha, \tau) \) be the collapse of \( H_\nu(\alpha, \tau) \). Let \( \tau_0 \) be the minimal \( \tau \) such that \( \nu^*, \alpha^*_{\mu}, \omega^*_{\mu}, P^*_{\nu} \in S_\tau \). Define by induction for \( \tau_0 \leq \tau < \rho(\nu) \):

\[ \alpha(\tau_0) = \alpha_\nu \]

\[ \alpha(\tau + 1) = \text{sup}(M_\nu(\alpha(\tau), \tau + 1) \cap \nu) \]

\[ \alpha(\lambda) = \text{sup}\{ \alpha(\tau) \mid \tau < \lambda \} \text{ if } \lambda \in \text{Lim}. \]

Set

\[ B_\nu = \{ \langle \alpha(\tau), M_\nu(\alpha(\tau), \tau) \rangle \mid \tau_0 < \tau < \rho(\nu) \} \text{ if } \nu < \rho(\nu), \]

\[ B_\nu = \{ 0 \} \times A_\nu \cup \{ \{ 1, \nu^*, \alpha^*_{\mu}, \omega^*_{\mu}, P^*_{\nu} \} \} \text{ else.} \]

Lemma 22

\[ B_\nu \subseteq J^X_\nu \text{ and } \langle I^0_\nu, B_\nu \rangle \text{ is rudimentary closed.} \]

**Proof:** If \( \nu = \rho(\nu) \), then both claims are clear. Otherwise, we first prove \( M^\nu(\alpha, \tau) \in J^X_\nu \) for all \( \alpha < \nu \) and all \( \tau \in \rho(\nu) \) such that \( \tau_0 \leq \tau < \rho(\nu) \). Let such a \( \tau \) be given and \( \tau' \in \rho(\nu) - \text{Lim} \) be such that \( X \cap S_\tau, A_\nu \cap S_\tau \subseteq S_{\tau'} \) (rudimentary closedness of \( \langle I^0_\nu, A_\nu \rangle \)). Let \( H := \text{sup}(\tau' \cap \text{Lim}) \). Let \( H \) be the closure of \( \alpha \cup \{ \nu^*, \alpha^*_{\mu}, \omega^*_{\mu}, P^*_{\nu}, X \cap S_\tau, A_\nu \cap S_\tau, \eta \} \) under all \( h^\varphi_{\tau',S_\tau \cap A_\nu \cap S_\tau} \). Let \( \sigma : H \cong S \) be the collapse of \( H \) and \( \sigma(\eta) = \bar{\eta} \). If \( \eta \in S^X \), then \( S = S_{\tau'} \) for some \( \tau' \) by the condensation property of \( L[X] \). If \( \eta \notin S^X \), then \( S = S^X[\eta] \) for some \( \bar{\eta} \) where \( S^X[\eta] \) is defined like \( S_{\tau'} \) with \( X \upharpoonright \bar{\eta} \) instead of \( X \). The reason is that, even if \( \eta \notin S^X \), it is the supremum of points in \( S^X \), because \( S^X = \{ \beta(\nu) \mid \nu \text{ singular in } L_\nu[X] \} \). In both cases, \( S \in J^X_\nu \) and there is a function in \( I^\nu_{\eta+\omega} \) that maps \( \alpha \cup \{ \sigma(\nu^*), \sigma(\alpha^*_{\nu}), \sigma(\alpha^*_{\mu}), \sigma(\omega^*_{\mu}), \sigma(P^*_{\nu}), \sigma(X \cap S_\tau), \sigma(S_\tau), \sigma(A_\nu \cap S_\tau), \sigma(\eta) \} \) onto \( S \). So \( \nu \) would be singular in \( J^X_\nu \) if \( \nu \leq \tau' \). But this contradicts the definition of \( \beta(\nu) \). Therefore, \( \sigma(\nu^*), \sigma(\alpha^*_{\nu}), \sigma(\alpha^*_{\mu}), \sigma(\omega^*_{\mu}), \sigma(P^*_{\nu}), \sigma(X \cap S_\tau), \sigma(S_\tau), \sigma(A_\nu \cap S_\tau), \sigma(\eta) \} \) under all \( h^\varphi_{\tau',S_\tau \cap A_\nu \cap S_\tau} \) where these are defined like \( h^\varphi_{\nu,E} \), but with \( \sigma(S_\tau) \) instead of \( S_\tau \). Then \( H_\nu(\alpha, \tau) \preceq \langle S_\tau, \sigma(\nu^*), \sigma(\alpha^*_{\nu}), \sigma(\alpha^*_{\mu}), \sigma(\omega^*_{\mu}), \sigma(P^*_{\nu}), \sigma(X \cap S_\tau), \sigma(S_\tau), \sigma(A_\nu \cap S_\tau), \sigma(\eta) \rangle \) and \( M_\nu(\alpha, \tau) \) is the collapse of \( H_\nu(\alpha, \tau) \). Since \( \nu < \rho(\nu) \) and \( \nu \) is a cardinal in \( I^\nu_{\beta(\nu)}, J^X_\nu \models ZF^- \). So we can form the collapse inside \( J^X_\nu \). Thus \( M_\nu(\alpha, \tau) \in J^X_\nu \).

Now, we turn to rudimentary closedness. Since \( B_\nu \) is unbounded in \( \nu \), it suffices to prove that the initial segments of \( B_\nu \) are elements of \( J^X_\nu \). Such an initial segment is of the form \( \langle M_\nu(\alpha(\tau), \gamma) \mid \gamma < \rho(\nu) \rangle \), and we have \( H_\nu(\alpha(\tau), \delta_\tau) = H_\nu(\alpha(\tau), \tau) \) where \( \delta_\tau \) is for \( \tau < \gamma \) the least \( \eta \geq \tau \) such that \( \eta \in H_\nu(\alpha(\tau), \gamma) \cup \{ \gamma \} \). Since \( \delta_\tau \in H_\nu(\alpha(\tau), \gamma) \preceq \langle S_\gamma, X \cap S_\gamma, A_\nu \cap S_\gamma, \{ \ldots \} \rangle \),
\((H_\nu(\alpha(\tau), \delta_\tau))^{H_\nu(\alpha(\gamma), \gamma)} = H_\nu(\alpha(\gamma), \gamma)\). Let \(\pi : M_\nu(\alpha(\gamma), \gamma) \to S_\gamma\) be the un-collaps of \(H_\nu(\alpha(\gamma), \gamma)\). Then, by the \(\Sigma_1\)-elementarity of \(\pi\), \(M_\nu(\alpha(\tau), \tau) = M_\nu(\alpha(\tau), \delta_\tau)\) is the collapse of \((H(\alpha(\tau), \pi^{-1}(\delta_\tau)))^{M_\nu(\alpha(\gamma), \gamma)}\). So \(\langle M_\nu(\alpha(\tau), \tau) \mid \tau < \gamma\rangle\) is definable from \(M_\nu(\alpha(\gamma), \gamma) \in J_\nu\). \(\square\)

**Lemma 23**

For \(x, y_1 \in J_\nu^X\), the following are equivalent:

(i) \(x \in \Sigma_1\)-definable in \(\langle I_\rho(\nu), A_\nu \rangle\) with the parameters \(y_1, \nu^*, \alpha^*_\nu, p_\nu, \alpha^{**}_\nu, P^*_\nu\).

(ii) \(x \in \Sigma_1\)-definable in \(\langle I_0, B_\nu \rangle\) with the parameters \(y_1\).

**Proof:** For \(\nu = \rho(\nu)\), this is clear. Otherwise, let first \(x\) be uniquely determined in \(\langle I_\rho(\nu), A_\nu \rangle\) by \((\exists z)\psi(z, x, (y_1, \nu^*, \alpha^*_\nu, p_\nu, \alpha^{**}_\nu, P^*_\nu))\) where \(\psi\) is a \(\Sigma_0\) formula. That is equivalent to \((\exists \tau)(\exists z \in S_\nu)\psi(z, x, (y_1, \nu^*, \alpha^*_\nu, p_\nu, \alpha^{**}_\nu, P^*_\nu))\) and again to \((\exists \tau)H_\nu(\alpha(\tau), \tau) \models (\exists z)\psi(z, x, (y_1, \nu^*, \alpha^*_\nu, p_\nu, \alpha^{**}_\nu, P^*_\nu))\). If \(\tau\) is large enough, the \(y_1\) are not moved by the collapsing map, since then \(y_1 \in J_{\alpha(\tau)}^X \subseteq H_\nu(\alpha(\tau), \tau)\). Let \(\tilde{\nu}, \alpha, p, \nu^*, P\) be the images of \(\nu^*, \alpha^*_\nu, p_\nu, \alpha^{**}_\nu, P^*_\nu\) under the collapse. Then \((\exists \tau)(y_1 \in J_{\alpha(\tau)}^X\) and \(M_\nu(\alpha(\tau), \tau) \models (\exists z)\psi(z, x, (y_1, \tilde{\nu}, \alpha, p, \nu^*, P))\) defines \(x\). So it is definable in \(\langle I_0, B_\nu \rangle\).

Since \(B_\nu\) and the satisfaction relation of \(\langle I_\rho(\nu), B_\nu \rangle\) are \(\Sigma_1\)-definable over \(\langle I_\rho(\nu), A_\nu \rangle\), the converse is clear. \(\square\)

**Lemma 24**

Let \(H \prec_1 \langle I_\rho(\nu), B_\nu \rangle\) for a \(\nu \in E\) and \(\pi : \langle I_\rho(\nu), B_\nu \rangle \to \langle I_\rho(\nu), B_\nu \rangle\) be the collapse of \(H\). Then \(\mu \in E\) and \(B = B_\mu\).

**Proof:** First, we extend \(\pi\) like in lemma 19. Let

\[ M = \{ x \in J_\rho^X \mid x \in \Sigma_1\text{-definable in } \langle I_\rho(\nu), A_\nu \rangle \text{ with parameters from } \text{rng}(\pi) \cup \{ p_\nu, \nu^*, \alpha^*_\nu, \alpha^{**}_\nu, P^*_\nu \} \}. \]

Then \(\text{rng}(\pi) = M \cap J_\mu^X\). For, if \(x \in M \cap J_\mu^X\), then there are by definition of \(M\) \(y_1 \in \text{rng}(\pi)\) such that \(x \in \Sigma_1\text{-definable in } \langle I_\rho(\nu), A_\nu \rangle\) with the parameters \(y_1\) and \(p_\nu, \nu^*, \alpha^*_\nu, \alpha^{**}_\nu, P^*_\nu\). Thus it is \(\Sigma_1\)-definable in \(\langle I_0, B_\nu \rangle\) with the \(y_1\) by lemma 23. Therefore, \(x \in \text{rng}(\pi)\) because \(y_1 \in \text{rng}(\pi) \prec_1 \langle I_\rho(\nu), B_\nu \rangle\). Let \(\tilde{\pi} : \langle I_\rho(\nu), A_\nu \rangle \to \langle I_0, B_\nu \rangle\) be the collapse of \(M\). Then \(\tilde{\pi}\) is an extension of \(\pi\), since \(M \cap J_0^X\) is an \(\epsilon\)-initial segment of \(M\) and \(\text{rng}(\tilde{\pi}) = M \cap J_\mu^X\). In addition, there is by lemma 19 a \(\Sigma_{\tilde{\pi}(\nu)}\)-elementary extension \(\tilde{\pi} : I_\beta \to I_{\tilde{\beta}(\nu)}\) such that \(\rho\) is the \((n(\nu) - 1)\)-th projection of \(I_\beta\) and \(A\) is the \((n(\nu) - 1)\)-th standard code of it. Let \(\tilde{\pi}(\rho) = p_\nu\) and \(\tilde{\pi}(\alpha) = \alpha^*_\nu\). And we have \(\tilde{\pi}(\mu) = \nu < \beta(\nu)\). In this case, \(x \in \text{rng}(\tilde{\pi})\) by the definition of \(\nu^*\). Since \(\tilde{\pi}\) is \(\Sigma_1\)-elementary, cardinals of \(J_\mu^X\) are mapped on cardinals of \(J_\nu^X\).

Assume \(\nu \in S^+\). Suppose there was a cardinal \(\tau > \alpha\) of \(J_\mu^X\). Then \(\pi(\tau) > \alpha_\tau\) was a cardinal in \(J_\nu^X\). But this is a contradiction.

Next, we note that \(\mu\) is \(\Sigma_{\tilde{\pi}(\nu)}\)-singular over \(I_\beta\). If \(\nu \in S^+\), then, by the definition of \(p_\nu\), \(J_\mu^X = h_{\rho, A}[\omega \times (\alpha \times \{ p \})]\) is clear. So there is an over \(\langle I_\rho(\nu), A\rangle\) \(\Sigma_1\)-definable function from \(\alpha\) colinear into \(\mu\). But since \(\rho\) is the \((n(\nu) - 1)\)-th projection and \(A\) is the \((n(\nu) - 1)\)-th code of it, this function is \(\Sigma_{\tilde{\pi}(\nu)}\)-definable over \(I_\beta\). Now, suppose \(\nu \notin S^+\). Let \(\lambda := sup(\pi[\mu])\). Since \(\lambda > \alpha^*_\nu\), there is a \(\gamma < \lambda\) such that

\[ sup(h_{\rho(\nu), A_\nu}[\omega \times (J_\mu^X \times \{ q_\nu \})] \cap \nu) \geq \lambda. \]
And since $rng(\pi)$ is cofinal in $\lambda$, there is such a $\gamma \in rng(\pi)$. Let $\gamma = \pi(\bar{\gamma})$. By the $\Sigma_1$-elementarity of $\bar{\pi}$, $\gamma < \mu$ and setting $\bar{\pi}(q) = q_\nu$ we have for every $\eta < \mu$

$$\langle I_\mu, A \rangle \models (\exists x \in J^X_\nu) (\exists h_{\rho, A}(i, (x, p)) > \eta).$$

Hence $h_{\rho, A}[\omega \times \{J^X_\nu \times \{p\}\}]$ is cofinal in $\mu$. This shows $\mu \in E$.

On the other hand, $\mu$ is $\Sigma_{n(\nu) - 1}$-regular over $I_\beta$ if $n(\nu) > 1$. Assume there was an over $I_\beta, \Sigma_{n(\nu) - 1}$-definable function $f$ and some $x \in \mu$ such that $f[x]$ was cofinal in $\mu$. I.e. $\forall y \in \mu, (\exists z \in x)(f(z) > y)$ would hold in $I_\beta$. Over $I_\beta$, $(\exists z \in x)(f(z) > y)$ is $\Sigma_{n(\nu) - 1}$. So it is $\Sigma_0$ over $(I^0_\nu, A)$. But then also $\forall y \in \mu, (\exists z \in x)(f(z) > y)$ is $\Sigma_0$ over $(I^0_\nu, A)$ if $\mu < \rho$. Hence it is $\Sigma_{n(\nu)}$ over $I_\beta$. But then the same would hold for $\bar{\pi}(x)$ in $I_{\beta(\nu)}$. This contradicts the definition of $n(\nu)$! Now, let $\mu = \rho$. Since $\alpha$ is the largest cardinal in $I_\mu$, we had in $f$ also an over $I_\beta$ $\Sigma_{n(\nu) - 1}$-definable function from $\alpha$ onto $\rho$ and therefore one from $\alpha$ onto $\beta$. But this contradicts lemma 21 and the fact that $\rho$ is the $(n(\nu) - 1)$-th projection of $\beta$. If $n(\nu) = 1$, then we get with the same argument that $\mu$ is regular in $I_\beta$.

The previous two paragraphs show $\beta = \beta(\mu)$ and $n(\mu) = n(\nu)$. We are done if we can also show that $\alpha = \alpha^*_\mu, \pi(\alpha^*_\mu) = \alpha^*_\nu, p = \mu, \pi(P^*_\mu) = P^*_\nu$, because $\bar{\pi}$ is $\Sigma_1$-elementary, $\bar{\pi}(h^X_{\pi(\tau)}, x, p_\nu) = h^X_{\pi(\tau)}(x_\tau)$ for all $\Sigma_1$ formulas $\varphi$ and $x_\tau \in S_\tau$.

For $\nu \in S^+$, $\alpha = \alpha_\nu$ was shown above. So let $\nu \notin S^+$. By the $\Sigma_1$-elementarity of $\bar{\pi}$, we have for all $\alpha \in \mu$

$$h_{\rho, A}[\omega \times \{J^X_\nu \times \{p\}\}] \ni \mu = \alpha \iff h_{\rho(\nu), A_\nu}[\omega \times \{J^X_{\pi(\alpha)} \times \{p_\nu\}\}] \ni \nu = \pi(\alpha).$$

The same argument proves $\pi(\alpha^*_\mu) = \alpha^*_\nu$. Finally, $p = p_\mu$ and $\pi(P^*_\mu) = P^*_\nu$ can be shown as in (5) in the proof of lemma 19. □

**Lemma 25**

Let $H \prec_1 (I^0_\nu, B_\nu)$ and $\lambda = \sup(H \cap \nu)$ for a $\nu \in E$. Then $\lambda \in E$ and $B_\nu \cap J^X_\nu = B_\lambda$.

**Proof:** Let $\pi_0 : (I^0_\mu, B_\mu) \rightarrow (I^0_\lambda, B_\lambda) \cap J^X_\lambda$ be the uncollapse of $H$ and let $\pi_1 : (I^0_\lambda, B_\lambda \cap J^X_\lambda) \rightarrow (I^0_\nu, B_\nu)$ be the identity. Since $L[X]$ has coherence, $\pi_0$ and $\pi_1$ are $\Sigma_1$-elementary. By lemma 18, $\pi_0$ is even $\Sigma_1$-elementary, because it is cofinal. To show $B_\lambda = B_\nu \cap J^X_\nu$, we extend $\pi_0$ and $\pi_1$ to $\pi_0 : (I^0_{\rho(\mu)}, A_\mu) \rightarrow (I^0_\rho, A)$ and $\pi_1 : (I^0_\rho, A) \rightarrow (I^0_{\rho(\nu)}, A_\nu)$ in such a way that $\pi_0$ is $\Sigma_1$-elementary and $\pi_1$ is $\Sigma_0$-elementary. Then we know from lemma 19 that $\rho$ is the $(n(\nu) - 1)$-th projection of $\beta$ and $A$ is the $(n(\nu) - 1)$-th code of it. So there is a $\Sigma_{n(\nu)}$-elementary extension of $\pi_0 : I_\beta \rightarrow I_\beta$. We can again use the argument from lemma 24 to show that $\lambda$ is $\Sigma_{n_{\nu(\nu) - 1}}$-regular over $I_\beta$. But on the other hand, $\lambda$ is as supremum of $H \cap On \Sigma_{n(\nu)}$-singular over $I_\beta$. From this, we conclude as in the proof of lemma 24 that $B_\lambda = B_\nu \cap J^X_\nu$.

First, suppose $\nu \in S^+$. Since $\alpha_\nu \in H \prec_1 (I^0_\nu, B_\nu)$, $\alpha_\nu < \lambda \leq \nu$. Since $I_\nu \models (\alpha_\nu$ is the largest cardinal), we therefore have $\lambda \notin Card$. In addition, $\alpha_\nu$ is the largest cardinal in $I_\lambda$. Assume $\tau$ was the next larger cardinal. Then $\tau$ was $\Sigma_1$-definable in $I_\mu$ with parameter $\alpha_\nu$ and some $\tau \in H$ and hence it was in $H$. By the $\Sigma_1$-elementarity of $\pi_0$, $\pi_0^{-1}(\tau) > \pi_0^{-1}(\alpha_\nu) = \alpha_\mu$ was also a cardinal in $I_\mu$. But this contradicts the definition of $\alpha_\mu$. 25
But now to $B_\nu = B_\nu \cap J^X_{\lambda}$. First, assume $\nu \notin S^+$. Let $\pi = \pi_1 \circ \pi_0 : \langle I^0_{\mu}, B_\mu \rangle \to \langle I^0_{\mu}, B_\nu \rangle$ and $\tilde{\pi} : \langle I^0_{\rho(\mu)}, A_\mu \rangle \to \langle I^0_{\rho(\mu)}, A_\nu \rangle$ be the extension constructed in the proof of lemma 24. Let $\gamma = \sup(\operatorname{rng}(\tilde{\pi}))$. Then $\tilde{\pi}' = \tilde{\pi} \cap (J^X_{\lambda} \times J^X_{\lambda}) : \langle I^0_{\rho(\mu)}, A_\mu \rangle \to \langle I^0_{\rho(\mu)}, A_\nu \rangle$ is $\Sigma_0$-elementary, by coherence of $L_\alpha[X]$, and cofinal. Thus $\tilde{\pi}'$ is $\Sigma_1$-elementary. Let $H' = h_{\gamma,A_\nu \cap J^X_{\lambda}}[\omega \times (J^X_{\lambda} \times \{p_\nu\})]$ and $\hat{\pi}_1 : \langle I^0_{\rho}, A \rangle \to \langle I^0_{\rho}, A \rangle$ be the uncollapse of $H'$. Then $H = \operatorname{rng}(\tilde{\pi}') \subseteq H'$. To see this, let $z = \operatorname{rng}(\tilde{\pi}')$ and $z = \tilde{\pi}'(y)$. Then by definition of $p_\nu$, there is an $x \in J^X_{\lambda}$ and an $i \in \omega$ such that $y = h_{\rho(\nu),A_\nu}(i, \langle x, p_\nu \rangle)$. By the $\Sigma_1$-elementarity of $\tilde{\pi}'$, we therefore have $z = h_{\gamma,A_\nu \cap J^X_{\lambda}}(i, \langle \tilde{\pi}'(x), \tilde{\pi}'(p_\nu) \rangle)$. But $\tilde{\pi}'(p_\nu) = \tilde{\pi}(p_\nu) = p_\nu$ and $\tilde{\pi}'(x) \in J^X_{\lambda}$.

In addition, $\sup(H' \cap \nu) = \lambda$. That $\sup(H' \cap \nu) \geq \lambda$ is clear. Conversely, let $x \in H' \cap \nu$, i.e. $x = h_{\gamma,A_\nu \cap J^X_{\lambda}}(i, \langle y, p_\nu \rangle)$ for some $i \in \omega$ and $y \in J^X_{\lambda}$. Then $x$ is uniquely determined by $\langle I^0_{\rho}, A \cap J^X_{\lambda} \rangle \models (\exists z)\psi_i(x, y, p_\nu)$. But such a $z$ exists already in a $H^\nu_0(\alpha, \tau)$ where $H^\nu_0(\alpha, \tau)$ is the closure of $S_\alpha$ under all $h^\nu_{X \cap n, A \cap J^X_{\lambda}}$. Since $\gamma = \sup(\operatorname{rng}(\tilde{\pi}))$ and $\lambda = \sup(\operatorname{rng}(\pi))$, we can pick such $\tau \in \operatorname{rng}(\tilde{\pi})$ and $\alpha \in \operatorname{rng}(\pi)$. Then $\tilde{\pi} = \pi_1^{-1}(\tau)$ and $\tilde{\alpha} = \pi_1^{-1}(\alpha)$. Let $\vartheta = \sup(\nu \cap H^\nu_0(\alpha, \tau))$ and $\tilde{\vartheta} = \sup(\mu \cap H^\nu_0(\alpha, \tau))$. Since $\nu$ is regular in $I^\nu(\nu)$, $\vartheta < \nu$. Analogously, $\vartheta < \mu$. But of course $\pi(\vartheta) = \vartheta$. So $x < \vartheta = \tilde{\pi}(\vartheta) < \sup(\pi(p_\nu)) = \lambda$.

If $\nu \in S^+$, we may define $H'$ as $h_{\gamma,A_\nu \cap J^X_{\lambda}}[\omega \times (J^X_{\lambda} \times \{p_\nu\})]$ and still conclude that $H = \operatorname{rng}(\tilde{\pi}') \subseteq H'$ and $\sup(H' \cap \nu) = \lambda$ by the definition of $p_\nu$.

By lemma 19, $\tilde{\pi} : \langle I^0_{\rho}, A \rangle \to \langle I^0_{\rho}, A \rangle$ may be extended to a $\Sigma_0(\nu)$-1-elementary embedding $\pi_1 : I_\beta \to I_{\beta(\nu)}$ such that $\rho$ is the $(\nu(\nu) - 1)$-th projection of $I_\nu$ and $A$ is the $(\nu(\nu) - 1)$-th standard code of it. Let $\tilde{\pi}_0 = \pi_1^{-1} \circ \tilde{\pi}$. Then $\tilde{\pi}_0 : \langle I^0_{\rho(\mu)}, A_\mu \rangle \to \langle I^0_{\rho(\mu)}, A_\nu \rangle$ is $\Sigma_0$-elementary, by the coherence of $L_\nu[X]$, and cofinal. Thus it is $\Sigma_1$-elementary by lemma 18. Applying again lemma 19, we get a $\Sigma_0(\nu)$-1-elementary embedding $\pi_0 : I_{\beta(\nu)} \to I_\beta$. As in lemma 24, it suffices to prove $\beta = \beta(\lambda)$, $\nu(\nu) = \lambda(\lambda)$, $\rho = \rho(\lambda)$, $A = A_\lambda$, $\pi_1^{-1}(p_\mu) = p_\lambda$, $\pi_1^{-1}(P^\nu_\mu) = P^\lambda_\mu$, $\alpha_\nu^* = \alpha_\lambda^*$ and $\pi_1^{-1}(\alpha_\nu^*) = \alpha_\mu^*$. So, if $\nu(\nu) > 1$, we have to show that $\lambda$ is $\Sigma_\nu(\nu)$-1-regular over $I_\beta$. If $\nu(\nu) = 1$, then $I_\beta \models (\lambda$ regular) suffices. In addition, $\lambda$ must be $\Sigma_\nu(\nu)$-singular over $I_\beta$. For regularity, consider $\tilde{\pi}_0$ and, as in lemma 24, the last $x$ in $\lambda$ proving the opposite if such an $x$ exists. This is again $\Sigma_\nu$-definable and therefore in $\operatorname{rng}(\tilde{\pi}_0)$. But then $\tilde{\pi}_0^{-1}(x)$ had the same property in $I_{\beta(\nu)}$. Contradiction!

Now, assume $\nu \in S^+$. Since $I_{\nu(\nu)} \models (\alpha_\nu$ is the largest cardinal), $H' \cap \nu$ is transitive. Thus $H'' \cap \nu = \lambda$. Since $\pi_1 : \langle I^0_{\rho}, A \rangle \to \langle I^0_{\rho}, A \cap J^X_{\lambda} \rangle$ is $\Sigma_1$-elementary and $\mu \setminus H'' = \operatorname{rng}(\tilde{\pi}_0)$, we have $\lambda = \lambda \cap h_{\rho(\nu),A}[\omega \times (J^X_{\lambda} \times \{\tilde{\pi}_0^{-1}(p_\nu)\})].$ I.e. there is a $\Sigma_1$-map over $\langle I_\nu, A \rangle$ from $\alpha_\nu$ onto $\lambda$. But this is then $\Sigma_{\nu(\nu)}$-definable over $I_\beta$ and $\lambda$ is $\Sigma_{\nu(\nu)}$-singular over $I_\beta$.

If $\nu \notin S^+$, then the fact that $\lambda$ is $\Sigma_{\nu(\nu)}$-singular over $I_\beta$, $\alpha_\nu^* = \alpha_\lambda^*$ and $\tilde{\pi}_0^{-1}(\alpha_\nu^*) = \alpha_\mu^*$ may be seen as in lemma 24 because $\pi_0(\alpha_\mu^*) = \alpha_\nu^* \in \operatorname{rng}(\pi_0)$.

That $\tilde{\pi}_0^{-1}(p_\mu) = p_\lambda$ and $\tilde{\pi}_0^{-1}(P^\nu_\mu) = P^\lambda_\mu$ can again be proved as in (5) in the proof of lemma 19. □

**Lemma 26**

Let $\nu \in E$ and $\Lambda(\xi, \nu) = \{\sup(h_{\rho(\nu),A}[\omega \times (J^X_{\beta} \times \{\xi\})] \cap \nu) \mid \beta \in \operatorname{Lim} \cap \nu\}$. Let $\check{\eta} < \check{\nu}$ and $\pi : \langle I^0_{\rho}, B \rangle \to \langle I^0_{\rho}, B_\nu \rangle$ be $\Sigma_1$-elementary. Then $\Lambda(\xi, \nu) \cap \check{\eta} \in J^X_{\rho}$ and $\pi(\Lambda(\xi, \nu) \cap \check{\eta}) = \Lambda(\xi, \nu) \cap \check{\eta}(\check{\eta})$ where $\check{\eta}(\check{\eta}) = \eta$. 26
Proof:

(1) Let $\lambda \in \Lambda(\xi, \nu)$. Then $\Lambda(\xi, \nu) = \Lambda(\xi, \nu) \cap \lambda$.

Let $\beta_0$ be minimal such that

$$\sup(h_{\nu, B_\lambda}[\omega \times (J^n_\lambda \times \{\xi\})] \cap \nu) = \lambda.$$ 

Then, by lemma 25, for all $\beta \leq \beta_0$

$$h_{\lambda, B_\lambda}[\omega \times (J^n_\beta \times \{\xi\})] = h_{\nu, B_\lambda}[\omega \times (J^n_\beta \times \{\xi\})]$$

and for all $\beta_0 \leq \beta$

$$h_{\lambda, B_\lambda}[\omega \times (J^n_\beta \times \{\xi\})] \subseteq h_{\lambda, B_\lambda}[\omega \times (J^n_\lambda \times \{\xi\})] \subseteq h_{\lambda, B_\lambda}[\omega \times (J^n_\lambda \times \{\xi\})].$$

So $\Lambda(\xi, \nu) = \Lambda(\xi, \nu) \cap \lambda$.

(2) Let $\lambda := \sup(\Lambda(\xi, \nu) \cap \eta + 1)$. Then, by (1), $\Lambda(\xi, \nu) \cap \eta + 1 = \Lambda(\xi, \nu) \cup \{\lambda\}$. But $\Lambda(\xi, \nu)$ is definable over $I_{\beta(\lambda)}$. Since $\beta(\lambda) < \nu$, we get $\Lambda(\xi, \nu) \cap \eta + 1 \in J^X_\nu$.

(3) Let $\sup(h_{\nu, B_\lambda}[\omega \times (J^n_\lambda \times \{\xi\})] \cap \nu) = \pi(\lambda)$. Then

$$\pi(\sup(h_{\nu, B_\lambda}[\omega \times (J^n_\lambda \times \{\xi\})] \cap \nu)) = \sup(h_{\nu, B_\lambda}[\omega \times (J^n_\lambda \times \{\xi\})] \cap \nu).$$

Let $\lambda := \sup(h_{\nu, B_\lambda}[\omega \times (J^n_\lambda \times \{\xi\})] \cap \nu)$. Then $(I^n_\nu, B_\nu) \models \neg(\exists \lambda \forall \theta)(\exists \bar{\xi} \in \omega)(\exists i < \beta)(\theta = h_{\nu, B_\nu}(i, \bar{\xi}, \bar{\xi}))$. So $(I^n_\nu, B_\nu) \models \neg(\exists \lambda < \theta)(\exists \bar{\xi} \in \omega)(\exists i < \beta)(\theta = h_{\nu, B_\nu}(i, \bar{\xi}, \bar{\xi}))$ where $\pi(\lambda) = \lambda$. i.e. $\sup(h_{\nu, B_\lambda}[\omega \times (J^n_\lambda \times \{\xi\})] \cap \nu) = \lambda$.

But $(\pi \mid J^X_\nu) : (I^n_\nu, B_\nu) \rightarrow (I^n_\nu, B_\lambda)$ is elementary. So, if $(I^n_\nu, B_\nu) \models (\forall n)(\exists \bar{\xi} \in \beta)(\exists \eta \in \omega)(\exists \lambda \in \nu)\exists \pi(\lambda) = \lambda \exists (\exists \lambda \in \nu)(\exists \eta \in \omega)(\exists \lambda \in \nu)\exists \pi(\lambda) = \lambda \exists (\exists \lambda \in \nu)(\exists \eta \in \omega)(\exists \lambda \in \nu)\exists \pi(\lambda) = \lambda$. But by lemma 25, $h_{\lambda, B_\lambda}[\omega \times (J^n_\lambda \times \{\xi\})] \subseteq h_{\nu, B_\lambda}[\omega \times (J^n_\lambda \times \{\xi\})]$. I.e. it is indeed $\lambda = \sup(h_{\nu, B_\lambda}[\omega \times (J^n_\lambda \times \{\xi\})] \cap \nu)$.

(4) Let $\lambda \in \Lambda(\xi, \nu)$.

For $\lambda \in \Lambda(\xi, \nu)$,

$$\pi(\Lambda(\xi, \nu) \cap \eta) = \Lambda(\xi, \nu) \cap \pi(\eta)$$

by (1) and (3).

$$\Lambda(\xi, \nu) \cap \pi(\eta)$$

by $\Sigma_1$-elementarity of $\pi$

$$\Lambda(\xi, \nu) \cap \pi(\eta)$$

by (1) and (3).

$$\Lambda(\xi, \nu) \cap \pi(\eta).$$

So, if $\Lambda(\xi, \nu)$ is cofinal in $\nu$, then we are finished. But if there exists $\lambda := \max(\Lambda(\xi, \nu))$, then, by (1) and (2), $\Lambda(\xi, \nu) \in J^X_\nu$, and it suffices to show $\pi(\Lambda(\xi, \nu)) = \Lambda(\xi, \nu)$. To this end, let $\beta$ be maximal such that $\lambda = \sup(h_{\nu, B_\lambda}[\omega \times (J^n_\lambda \times \{\xi\})] \cap \nu) = \lambda$. I.e. $h_{\nu, B_\lambda}[\omega \times (J^n_\lambda \times \{\xi\})] \subseteq h_{\nu, B_\lambda}[\omega \times (J^n_\lambda \times \{\xi\})]$. Hence indeed $\pi(\Lambda(\xi, \nu)) = \Lambda(\xi, \nu)$.

\[\square\]

Lemma 27

Let $\nu \in E, H \prec_1 (I^n_\nu, B_\nu)$ and $\lambda = \sup(H \cap \nu)$. Let $h : I^n_\nu \rightarrow I^n_\lambda$ be $\Sigma_1$-elementary and $H \subseteq \text{rng}(h)$. Then $\lambda \in E$ and $h : (I^n_\nu, B_\lambda) \rightarrow (I^n_\lambda, B_\lambda)$ is $\Sigma_1$-elementary.

Proof: By lemma 25, $B_\lambda = B_\nu \cap J^X_\lambda$. So it suffices, by lemma 24, to show

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rng(h) ≺_1 \langle I^0_\mu, B_\lambda \rangle. Let x_i \in \text{rng}(h) and \langle I^0_\mu, B_\lambda \rangle \models (\exists z)\psi(z, x_i) for a \Sigma_0 formula \psi. Then we have to prove that there exists a z \in \text{rng}(h) such that
\langle I^0_\mu, B_\lambda \rangle \models \psi(z, x_i). Since \lambda = \text{sup}(H \cap \nu), there is a \eta \in H \cap \text{Lim} such that \langle I^0_\eta, B_\lambda \cap J^X_\eta \rangle \models (\exists z)\psi(z, x_i). And since H \not\models_1 \langle I^0_\nu, B_\nu \rangle, we have \langle I^0_\eta, B_\lambda \cap J^X_\eta \rangle \in H \subseteq \text{rng}(h). So also
\text{rng}(h) \models (\langle I^0_\eta, B_\lambda \cap J^X_\eta \rangle \models (\exists z)\psi(z, x_i)) because \text{rng}(h) \not\models I^0_\mu. Hence there is a z \in \text{rng}(h) such that \langle I^0_\eta, B_\lambda \cap J^X_\eta \rangle \models \psi(z, x_i). I.e. \langle I^0_\mu, B_\lambda \rangle \models \psi(z, x_i). □

Lemma 28
Let f : \nu \Rightarrow \nu, \nu \sqsubseteq \mu_\rho and f(\bar{\tau}) = \tau. If \bar{\tau} \in S^+ \cup \hat{S} is independent, then (f \upharpoonright J^D_{\alpha_\tau}) : \langle J^D_{\alpha_\tau}, D_{\alpha_\tau}, K_{\bar{\tau}} \rangle \rightarrow \langle J^D_{\alpha_\tau}, D_{\alpha_\tau}, K_{\bar{\tau}} \rangle is \Sigma_1\text{-elementary.}

Proof: If \bar{\tau} = \mu_\rho < \mu_\nu, then the claim holds since | f | : I_{\mu_\rho} \rightarrow I_{\mu_\nu} is \Sigma_1\text{-elementary. If } \mu_\tau = \mu_\nu and n(\tau) = n(\nu), then P_{\tau} \subseteq P_{\nu}. I.e. \tau is dependent on \nu. Thus \bar{\tau} is not independent. So let \mu := \mu_\tau = \mu_\nu, n := n(\tau) < n(\nu) and \tau \in S^+ \cup \hat{S} be independent. Then, by the definition of the parameters, \alpha_\tau is the n-th projection of \mu.

Let
\gamma_\beta := \text{crit}(f(\beta, 0, \tau)) < \alpha_\tau
for a \beta and
H_\beta := \text{the } \Sigma_\mu\text{-hull of } \beta \cup P_\tau \cup \{\alpha_\beta^*, \tau\}\text{ in } I_{\mu}.
I.e. H_\beta = h^\mu_\beta[\omega \times (J^X_\beta \times \{\alpha_\beta^*, \tau'\}, P^\tau_\beta)] where
\alpha_\beta^* := \text{minimal such that } h^\mu_\beta(i, \alpha_\beta^*) = \alpha_\beta^* \text{ for an } i \in \omega
P^\tau_\beta := \text{minimal such that } h^\mu_\beta(i, P^\tau_\beta) = P_\tau \text{ for an } i \in \omega
\tau' := \text{minimal such that } h^\mu_\beta(i, \tau') = \tau' \text{ for an } i \in \omega \text{ (resp. } \tau' := 0 \text{ for } \tau = \mu).

For the standard parameters are in P_\tau.
so H_\beta is \Sigma_\mu\text{-definable over } I_{\mu} with the parameters \{\beta, \tau, \alpha_\beta^*, P_\tau\}. Let
\rho := \alpha_\tau = \text{the } n\text{-th projection of } \mu
A := \text{the } n\text{-th standard code of } \mu
p := (\alpha_\beta^*, \tau', P^\tau_\beta).
So H_\beta \cap J^X_\rho is \Sigma_\mu\text{-definable over } \langle I^0_\mu, A \rangle with parameters \beta and \rho. (fine structure theory!)
And \gamma_\beta is defined by
\gamma_\beta \not\in H_\beta \text{ and } (\forall \delta \in \gamma_\beta)(\delta \in H_\beta).
I.e. \gamma_\beta is also \Sigma_\mu\text{-definable over } \langle I^0_\mu, A \rangle with parameters \beta and \rho.

Let f_0 := f(\beta, 0, \tau) for a \beta, \bar{\tau}_0 := d(f_0) < \alpha_\tau and \gamma := \text{crit}(f_0) < \alpha_\tau. Let f_1 := f(\beta, \gamma, \tau), \bar{\tau}_1 := d(f_1) < \alpha_\tau and \delta := \text{crit}(f_1) < \alpha_\tau. Then \mu_{\tau_1} is the direct successor of \mu_{\tau_0} in K_\tau. So f(\beta, \gamma, \tau_1) = \text{id}_{\tau_1}. Hence \mu_\eta = \mu_{\tau_1} holds for the minimal \eta \in S^+ \cup S^0 such that \gamma < \eta \subseteq \delta. Thus
\mu' \in K^+_\tau := K_\tau - (\text{Lim}(K_\tau) \cup \{\text{min}(K_\tau)\})
\iff

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\((\exists \beta, \gamma, \delta, \eta) (\gamma = \gamma_0 \text{ and } \delta = \gamma (\gamma_0 + 1))\)

and \(\eta \in S^+ \cup S^0\) minimal such that \(\gamma < \eta \subseteq \delta\) and \(\mu' = \mu_\eta\)

Therefore, \(K^\gamma_\gamma\) is \(\Sigma_1\)-definable over \((I^0_\mu, A)\) with parameter \(p\).

Now, consider \((I^0_{\alpha_\gamma}, K_\gamma) \models \varphi(x)\) where \(\varphi\) is a \(\Sigma_1\) formula. Then, since \(K_\gamma\) is unbounded in \(\alpha_\gamma\),

\[ (I^0_{\alpha_\gamma}, K_\gamma) \models \varphi(x) \]

\[
\iff \\
(\exists \gamma)(\gamma \in K^\gamma_\gamma \text{ and } (I^0_{\alpha_\gamma}, K_\gamma) \models \varphi(x)). \]

So \((I^0_{\alpha_\gamma}, K_\gamma) \models \varphi(x)\) is \(\Sigma_1\) over \((I^0_\mu, A)\) with parameter \(p\), rso. \(\Sigma_{n+1}\) over \(I_\mu\) with parameters \(\alpha_\mu^0, \tau, P_\gamma\). But since \(n = n(\tau) < n(\nu)\), \(f\) is at least \(\Sigma_{n+1}\)-elementary. In addition \(f(\alpha_\mu^0) = \alpha_\mu^0\), \(f(\check{r}) = \tau, f(P_\gamma) = P_\gamma\). So, for \(x \in \text{rng}(f)\),

\((I^0_{\alpha_\gamma}, K_\gamma) \models \varphi(f^{-1}(x))\) holds iff \((I^0_{\alpha_\gamma}, K_\gamma) \models \varphi(x)\). \(\square\)

Theorem 29

\(\mathfrak{M} := \langle S, \prec, \mathcal{S}, D \rangle\) is a \(\kappa\)-standard morass.

Proof: Set

\[ \sigma_{(\xi, \nu)}(i) = h^{(\nu)}_{\sigma}(i, \langle \xi, \alpha_\mu^0, p_\nu \rangle). \]

Then \(D\) is uniquely determined by the axioms of standard morasses and

1. \(D_\nu\) is uniformly definable over \(\langle J^X_\nu, X \upharpoonright \nu, X_\nu \rangle\)
2. \(X_\nu\) is uniformly definable over \(\langle J^D_\nu, D_\nu, D_\nu \rangle\).

(1) is clear. For (2), assume first that \(\nu \in \check{S}\) and \(f(0, q_\nu, \nu) = \nu_\nu\). Since the set \(\{ i \mid \sigma_{(q_\nu, \nu)}(i) \in X_\nu \}\) is \(\Sigma_{n(\nu)}\)-definable over \(\langle J^X_\nu, X \upharpoonright \nu, X_\nu \rangle\) with the parameters \(p_\nu, \alpha_\nu^0, q_\nu\), there is a \(j \in \omega\) such that

\[ \sigma_{(q_\nu, \nu)}(\langle i, j \rangle) \text{ existiert } \Leftrightarrow \sigma_{(q_\nu, \nu)}(i) \in X_\nu. \]

Using this \(j\), we have

\[ X_\nu = \{ \sigma_{(q_\nu, \nu)}(i) \mid \langle i, j \rangle \in \text{dom}(\sigma_{(q_\nu, \nu)}) \}. \]

So, in case that \(f(0, q_\nu, \nu) = \nu_\nu\), there is the desired definition of \(X_\nu\).

Let \(\nu \in \check{S}\), \(f(0, q_\nu, \nu) : \nu \Rightarrow \nu\) cofinal and \(f(\check{q}) = q_\nu\). Then \(f(0, \check{q}, \nu) = \nu_\nu\). And by lemma 6 (b) of [Ir2], \(\check{q} = q_\nu\). So, if \(\check{\nu} = \nu\), then \(f(0, q_\nu, \nu) = \nu_\nu\). Thus let \(\check{\nu} < \nu\).

Then \(f_0(0, q_\nu, \nu)(x) = y\) is defined by: There is a \(\nu \leq \nu\) such that, for all \(r, s \in \omega\),

\[ \sigma_{(q_\nu, \nu)}(r) \leq \sigma_{(q_\nu, \nu)}(s) \Leftrightarrow \sigma_{(q_\nu, \nu)}(r) \leq \sigma_{(q_\nu, \nu)}(s) \]

holds and for all \(z \in J^X_\nu\) there is an \(s \in \omega\) such that

\[ z = \sigma_{(q_\nu, \nu)}(s) \]

\(\text{and there is an } s \in \omega\) such that

\[ \sigma_{(q_\nu, \nu)}(s) = x \Leftrightarrow \sigma_{(q_\nu, \nu)}(s) = y \]

\[ . \]
And since \( \langle J^X_\nu, X_\nu \rangle \) is rudimentary closed,

\[
X_\nu = \bigcup \{ f(X_\nu \cap \eta) \mid \eta < \bar{\nu} \}.
\]

Finally, if \( \nu \in \hat{S} \) and \( f_{(0, \eta, \nu)} \) is not cofinal in \( \nu \), then \( C_\nu \) is unbounded in \( \nu \) and

\[
X_\nu = \bigcup \{ X_\lambda \mid \lambda \in C_\nu \}
\]

by the coherence of \( L_\kappa [X] \).

So (2) holds. From this, \((DF)^+\) follows.

By (1) and (2), \( J^X_\nu = J^D_\nu \) for all \( \nu \in \text{Lim} \), and for all \( H \subseteq J^X_\nu = J^D_\nu \)

\[
H \prec_1 \langle J^X_\nu, X \mid \nu \rangle \iff H \prec_1 \langle J^D_\nu, D_\nu \rangle.
\]

Now, we check the axioms.

(MP) and \((MP)^+\)

\[ | f_{(0, \xi, \nu)} | \text{ is the uncollapse of } h^{\mu[\nu]}_{\mu[\nu]}[\omega \times \{ \xi^*, \nu^*, \alpha_{\mu}^{**}, \alpha_{\mu}^*, P_{\mu}^* \}^{<\omega}] \text{ where } \xi^* \text{ is minimal such that } h^{\mu[\nu]}_{\mu[\nu]}(i, \xi^*) = \xi. \]

Therefore, (MP) and \((MP)^+\) hold.

(LP1)

holds by (2) above.

(LP2)

This is lemma 26.

(CP1) and \((CP1)^+\)

This follows from lemma 24 and the definition of \( \sigma(\xi, \nu) \).

(CP2)

This is lemma 27.

(CP3) and \((CP3)^+\)

Let \( x \in J^X_\nu \), \( i \in \omega \) and \( y = h_{\nu, B_\nu}(i, x) \). Since \( C_\nu \) is unbounded in \( \nu \), there is a \( \lambda \in C_\nu \) such that \( x, y \in J^X_\lambda \). By lemma 25, \( B_\lambda = B_\nu \cap J^X_\lambda \). So \( y = h_{\lambda, B_\nu}(i, x) \).

(DP1)

holds by the definition of \( \mu_\nu \).

(DF)

Let \( \mu := \mu_\nu \), \( k := n(\mu) \) and

\[
\pi(n, \beta, \xi) := \text{the uncollapse of } h^{k+n}_{\mu}[\omega \times (J^X_\beta \times \{ \alpha_{\mu}^{**}, p_{\mu}^*, \xi^* \}^{<\omega})]
\]

where

\[
\xi^* := \text{minimal such that } h^{k+n-1}_{\mu}(i, \xi^*) = \xi \text{ for an } i \in \omega
\]

\[
p_{\mu}^* := \text{minimal such that } h^{k+n-1}_{\mu}(i, p_{\mu}^*) = p_{\mu} \text{ for some } i \in \omega
\]

\[
\alpha_{\mu}^{**} := \text{minimal such that } h^{k+n-1}_{\mu}(i, \alpha_{\mu}^{**}) = \alpha_{\mu}^* \text{ for some } i \in \omega.
\]

Prove

\[
| f_{(\beta, \xi, \mu)}^{1+n} | = \pi(n, \beta, \xi)
\]

for all \( n \in \omega \) by induction.

For \( n = 0 \), this holds by definition of \( f_{(\beta, \xi, \mu)}^1 = f_{(\beta, \xi, \mu)} \). So assume that |
Let \( \pi \) definition of \( \bar{\pi} \) and for a \( \bar{\pi} \) standard parameters including the \((k + m - 1)\)-th projectum of \( \mu \).

Let \( \pi(n, \beta, \xi) : I_\mu \to I_\mu \). Then
\[
(*) \quad \xi(m, \mu) = \pi(n, \beta, \xi) \xi(m, \bar{\mu}) \text{ for all } 1 \leq m \leq n;
\]

Let \( \pi(n, \beta, \xi), \alpha := \pi^{-1}[\alpha_{\tau(m, \mu)} \cap \text{rng}(\pi)], \rho := \pi(\alpha) \)
\[
r := \text{minimal such that } h_{\mu}^{k+m-2}(i, r) = p_\mu \text{ for an } i \in \omega
\]
\[
\alpha' := \text{minimal such that } h_{\mu}^{k+m-2}(i, \alpha') = \alpha_\mu^* \text{ for an } i \in \omega
\]

\( p := \text{the } (k + m - 1)\)-th parameter of \( \mu \)
and \( \pi(\bar{r}) = r, \pi(\bar{p}) = p, \pi(\bar{\alpha}') = \alpha' \). Let \( \xi := \xi(m, \bar{\mu}) \). Then \( \bar{p} = h_{\mu}^{k+m-1}(i, (\bar{x}, \bar{r}, \bar{\alpha}')) \)
for a \( \bar{x} \in J_\bar{X} \), because \( \alpha = \alpha_{\bar{\tau}(m, \bar{\mu})} \). So \( p = h_{\mu}^{k+m-1}(i, (x, \xi, r, \alpha')) \) where \( \pi(\bar{x}) = x \) and \( \pi(\bar{\xi}) = \xi \). Thus \( h_{\mu}^{k+m-1}[\omega \times (J_\bar{X}^{\bar{\alpha}(m, \bar{\mu})} \times \{\alpha', r, \xi \}^{<\omega}]) = J_\mu^X \)
by definition of \( p \). So \( \xi(m, \mu) \leq \xi \). Assume \( \xi(m, \mu) < \xi \). Then \( I_\mu \models (\exists \eta < \xi)(\exists i \in \omega)(\exists x \in J_\mu^X)(\xi = h_{\mu}^{k+m-1}(i, (x, \eta, \alpha'))) \). Or \( I_\mu \models (\exists \eta < \xi)(\exists i \in \omega)(\exists x \in J_\mu^X)(\xi = h_{\mu}^{k+m-1}(i, (\eta, x, \alpha'))) \).

But this contradicts the definition of \( \xi = \xi(m, \bar{\mu}) \).

So, for all \( 1 \leq m \leq n \),
\[
\xi(m, \mu) \in \text{rng}(\pi(n, \beta, \xi)).
\]

In addition, for all \( \beta < \alpha_{\tau(m, \mu)} \),
\[
d(f_{\beta, \xi, \mu}) < \alpha_{\tau(m, \mu)}.
\]

Consider \( \pi := \pi(m - 1, \beta, \xi) =: f_{\beta, \xi, \mu} \) where \( \xi = \xi(m, \mu) \). Then \( \pi : I_\mu \to I_\mu \)
is the uncollapse of \( h_{\mu}^{k+m-1}[\omega \times (\beta \times \{\xi, \alpha', r\}^{<\omega})] \) where
\[
r := \text{minimal such that } h_{\mu}^{k+m-2}(i, r) = p_\mu \text{ for some } i \in \omega
\]
\[
\alpha' := \text{minimal such that } h_{\mu}^{k+m-2}(i, \alpha') = \alpha_\mu^* \text{ for some } i \in \omega.
\]
And \( h_{\mu}^{k+m-1}[\omega \times (\beta \times \{\xi, \alpha', r\}^{<\omega})] = J_\mu^{X} \) where \( \pi(\bar{\xi}) = \xi \), \( \pi(\bar{\alpha}') = \alpha' \) and \( \pi(\bar{r}) = r \). Assume \( \alpha_{\tau(m, \mu)} \leq \bar{\mu} < \mu \). Then there were a function over \( I_\mu \) from \( \beta < \alpha_{\tau(m, \mu)} \) onto \( \alpha_{\tau(m, \mu)} \). This contradicts the fact that \( \alpha_{\tau(m, \mu)} \) is a cardinal in \( I_{\bar{\mu}} \). If \( \bar{\mu} = \mu \), then \( f_{\beta, \xi, \mu} = \text{id}_\mu \). This contradicts the minimality of \( \tau(m, \mu) \).

Since \( \xi(m, \mu) \in \text{rng}(\pi(n, \beta, \xi)) \), we can prove
\[
\text{rng}(\pi(n, \beta, \xi)) \cap J_{\alpha_{\tau(m, \mu)}}^D \preceq_1 (J_{\alpha_{\tau(m, \mu)}}^D, D_{\alpha_{\tau(m, \mu)}}^D, K_{\mu}^m)
\]
for all \( 1 \leq m \leq n \) as in lemma 28.

We still must prove minimality. Let \( f \Rightarrow \mu \) and \( \beta \cup \{\xi\} \leq \text{rng}(f) \) such that
\[
\text{rng}(f) \cap J_{\alpha_{\tau(m, \mu)}}^D \preceq_1 (J_{\alpha_{\tau(m, \mu)}}^D, D_{\alpha_{\tau(m, \mu)}}^D, K_{\mu}^m)
\]
\[
\xi(m, \mu) \in \text{rng}(f)
\]
holds for all \( 1 \leq m \leq n \). Show that \( f \) is \( \Sigma_{k+n} \)-elementary and that the first standard parameters including the \((k + n - 1)\)-th are in \( \text{rng}(f) \). That suffices because \( \pi(n, \beta, \xi) \) is minimal.

Let \( p_{\mu}^{k+m} \) be the \((k + m)\)-th standard parameter of \( \mu \).
Prove, by induction on $0 \leq \alpha$.

$f$ is $\Sigma_{k+m}$-elementary

$$p_0^1, \ldots, p_{k+m}^{k+m-1} \in \text{rng}(f).$$

For $m = 0$, this is clear because $f \Rightarrow \mu$. So assume it to be proved for $m < n$ already. Then let $\alpha := \alpha + (m+1, \mu)$ and $\beta = f^{-1}[\alpha \cap \text{rng}(f)]$. Consider $\pi := (f \upharpoonright J^D_\alpha : \langle J^D_\alpha, D_\alpha, K_\mu \rangle \to \langle J^D_\alpha, D_\alpha, K_\mu^{m+1} \rangle)$. Construct a $\Sigma_{k+m+1}$-elementary extension $\tilde{\pi}$ of $\pi$. To do so, set

$$f_\beta = f_{(3,\xi(m+1,\mu),\nu)}^{m+1}$$

$$\mu(\beta) = d(f_\beta)$$

$$H = \bigcup \{ f_\beta[\text{rng}(\pi) \cap J^D_{\mu(\beta)}] \mid \beta < \alpha \}.$$ Then $H \cap J^D_\alpha = \text{rng}(\pi)$. For, $\text{rng}(\pi) \subseteq H \cap J^D_\alpha$ is clear because $f_\beta \upharpoonright J^D_\alpha = id \upharpoonright J^D_\beta$. So let $y \in H \cap J^D_\alpha$. I.e. $y = f_\beta(x)$ for some $x \in \text{rng}(\pi)$ and a $\beta \prec \alpha$. Let $K^+ = K^{m+1} - \text{Lim}(K_\mu^{m+1})$ and $\beta(\eta) = \sup \{ \beta \mid f_{(\beta,\xi(m+1,\eta),\nu)}^{m+1} \neq \eta \}$. Then

$$\langle J^D_\alpha, D_\alpha, K_\mu^{m+1} \rangle = (\exists y)(\exists \xi \in K^+)(y = f_{(\beta,\xi(m+1,\eta),\nu)}^{m+1}(x) \in J^D_\beta(\eta)).$$

Since $\text{rng}(\pi) \prec_1 \langle J^D_\alpha, D_\alpha, K_\mu^{m+1} \rangle$, $y = f_{(\beta,\xi(m+1,\eta),\nu)}^{m+1}(x) \in \text{rng}(\pi)$ if $x \in \text{rng}(\pi)$ for such an $\eta$. But since $y = f_{(\beta,\xi(m+1,\eta),\nu)}^{m+1}(x) \in J^D_\beta(\eta)$, we get $f_\beta(x) = f_{(3,\xi(m+1,\eta),\nu)}^{m+1}(x) \in \text{rng}(\pi)$.

Show $H \prec_{k+m+1} I_\mu$. Since $f_{(3,\xi(m+1,\eta),\nu)}^{m+1} = \pi(m, \beta, \xi), \alpha + (m+1, \mu)$ is the $(k+m)$-th projectum of $\mu$. Like in $(*)$ above, we can show that the $(k+m)$-th standard parameter $p_{k+m}$ of $\mu$ is in $\text{rng}(f_\beta)$. Now, let $I_\mu := (\exists x)(\exists y)(p_1^1, \ldots, p_{k+m}^1) (x, y, p_1^1, \ldots, p_{k+m}^1)$ where $\varphi$ is a $\Pi_{k+m}$ formula and $y \in H \cap J^D_\alpha$. Since $f_\beta$ is $\Sigma_{k+m}$-elementary, the following holds:

$I_\mu \models (\exists x)(\exists y)(p_1^1, \ldots, p_{k+m}^1) \Leftrightarrow (\exists \gamma \in K^{m+1}_\mu)(\exists \gamma')(I_\gamma \models \varphi(x, y, p_1^1, \ldots, p_{k+m}^1)).$

And since $\text{rng}(\pi) \prec_1 \langle J^D_\alpha, D_\alpha, K_\mu^{m+1} \rangle$,

$$\text{rng}(\pi) \models (\exists \gamma \in K^{m+1}_\mu)(\exists \gamma')(I_\gamma \models \varphi(x, y, p_1^1, \ldots, p_{k+m}^1)).$$

Thus there is such an $x$ in $\text{rng}(\pi)$ and therefore in $H$.

Let $\tilde{\pi}$ be the uncollapse of $H$. Then $\tilde{\pi}$ is $\Sigma_{k+m}$-elementary and, since $p_0^1, \ldots, p_{k+m}^1 \in \text{rng}(f_\beta)$ for all $\beta \prec \alpha$, we have $p_0^1, \ldots, p_{k+m}^1 \in \text{rng}(\pi) = H$. In addition, by the induction hypothesis, $f$ is $\Sigma_{k+m}$-elementary and $p_0^1, \ldots, p_{k+m-1}^1 \in \text{rng}(f)$.

Again as in $(*)$ above, we can show that $p_{k+m}^1 \in \text{rng}(f)$ using $\xi(m+1, \mu) \in \text{rng}(f)$. But since $\tilde{\pi}$ and $f$ are the same on the $(k+m)$-th projectum, we get $\tilde{\pi} = f$.

(SP) follows from $| f_{(3,\xi(m,\mu)}^{m+1} = \pi(n, \beta, \xi) $, because for all $\nu \subseteq \tau \subseteq \mu_\nu$ such that $\tau \in S^+$ (r.p. $\tau = \nu$) the following holds:

$$p_\tau \in \text{rng}(\pi(n, \beta, \xi)) \Leftrightarrow \xi_\tau \in \text{rng}(\pi(n, \beta, \xi)).$$

This may again be shown as $(*)$.  

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is like (∗) in (DF).

(DP3)

(a) is clear.

(b) was already proved with (DF)+.

□

**Theorem 30**

Let \( \langle X_\nu \mid \nu \in S^X \rangle \) be such that

1. \( L[X] \models S^X = \{ \beta(\nu) \mid \nu \text{ singular} \} \)
2. \( L[X] \) is amenable
3. \( L[X] \) has condensation
4. \( L[X] \) has coherence.

Then there is a sequence \( C = \langle C_\nu \mid \nu \in \hat{S} \rangle \) such that

1. \( L[C] = L[X] \)
2. \( L[C] \) has condensation
3. \( C_\nu \) is club in \( J^C_\nu \) w.r.t. the canonical well-ordering \( <_\nu \) of \( J^C_\nu \)
4. \( \text{otp}(C_\nu, <_{\nu}) > \omega \Rightarrow C_\nu \subseteq \nu \)
5. \( \mu \in \text{Lim}(C_\nu) \Rightarrow C_\mu = C_\nu \cap \mu \),
6. \( \text{otp}(C_\nu) < \nu \).

**Proof:** First, construct from \( L[X] \) a standard morass as in theorem 29. Then construct a inner model \( L[C] \) from it as in [Irr2]. □

**References**


[Irr2] B. Irrgang: *Proposing \((\omega_1, \beta)\)-Morasses*, \( \omega_1 \leq \beta \)
